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On the Nonexistence of Quaternary [51, 4, 37] Codes

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In this paper we prove the nonexistence of quaternary linear codes with parameters [51, 4, 37]. This result gives the exact value of $n_q(k, d)$ for $q = 4$, $k = 4$, $d = 37$ and 38. These were the only minimum distances for which the optimal length of a four-dimensional quaternary code was unknown. The proof is geometrical and relies heavily on results about the structure of certain sets of points in $PG(2, 4)$. © 1996 Academic Press, Inc.

1. INTRODUCTION

One of the central problems in coding theory is to determine the minimum possible length, denoted by $n_q(k, d)$, of a q-ary linear code of dimension k and minimum distance d. For quaternary codes, $n_4(k, d)$ was found for $k \leq 3$ for all d [1], and for $k = 4$ for all but two values of d [1], [3]. In this paper we prove the nonexistence of $[51, 4, 37]_4$ codes. This implies that $n_4(4, 37) = 52$ and $n_4(4, 38) = 53$, thus solving the remaining two cases of the problem for $k = 4$.

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We shall consider only codes which do not have any coordinate position where all the codewords have a zero entry. The columns of a generator matrix of such an $[n, k, d]_q$ code $\mathscr{C}$ can be considered as a multiset of n points in $PG(k-1, q)$ denoted by $\tilde{\mathscr{C}}$. Every hyperplane of $PG(k-1, q)$ meets $\tilde{\mathscr{C}}$ in at most $n - d$ points. In this paper we will consider codes entirely from this geometrical point of view. If the multiset $\tilde{\mathscr{C}}$ happens to be a set, we call it a *projective code.*

Given an $[n, k, d]_q$ code $\mathscr{C}$ we define $\tilde{\mathscr{C}}_\Delta = \{P \in \tilde{\mathscr{C}} | P \in \Delta\}$ and

$$\gamma_i(\tilde{\mathscr{C}}) = \max_{\Delta} |\tilde{\mathscr{C}}_\Delta|, \tag{1.1}$$

where Δ runs over all i-dimensional flats in $PG(k-1, q)$. In particular, $\gamma_0(\tilde{\mathscr{C}})$ is the maximum multiplicity of a point in $\tilde{\mathscr{C}}$. Often the code $\mathscr{C}$ will be clear from the context and we shall write simply γ_i.

The number of points in an i-flat is $(q^{i+1} - 1)/(q - 1)$, which we will denote by $\phi_q(i)$. We note also that the number of $(s-1)$-flats in $PG(k-1, q)$ containing a given $(s-2)$-flat is $\phi_q(k-s)$.

LEMMA 1.1. *Let $\mathscr{C}$ be an $[n, k, d]_q$ code, and let Π be an $(s-1)$-flat in $PG(k-1, q)$, $2 \le s < k$, meeting $\tilde{\mathscr{C}}$ in w points. Then for any $(s-2)$-flat Δ contained in Π, we have*

$$|\tilde{\mathscr{C}}_\Delta| \le \gamma_{s-1}(\tilde{\mathscr{C}}) - \frac{n - w}{\phi_q(k-s) - 1}. \tag{1.2}$$

In particular,

$$\gamma_{s-2}(\tilde{\mathscr{C}}) \le \gamma_{s-1}(\tilde{\mathscr{C}}) - \frac{n - \gamma_{s-1}(\tilde{\mathscr{C}})}{\phi_q(k-s) - 1}. \tag{1.3}$$

Proof. Counting the points of $\tilde{\mathscr{C}}$ lying in the $(s-1)$-flats containing Δ gives

$$w + (\phi_q(k-s) - 1)(\gamma_{s-1}(\tilde{\mathscr{C}}) - |\tilde{\mathscr{C}}_\Delta|) \ge n,$$

whence (1.2) follows. Now (1.3) follows since $\gamma_{s-1}(\tilde{\mathscr{C}})$ is the maximum value of w. ■

Consider an $[n, k, d]_q$ code $\mathscr{C}$ and denote by a_i the number of hyperplanes in the geometry $PG(k-1, q)$ containing exactly i points from $\tilde{\mathscr{C}}$, $i = 0, 1, \ldots, n-d$. Simple counting arguments yield the equalities

$$\sum_{i=0}^{n-d} a_i = \phi_q(k-1), \tag{1.4}$$

$$\sum_{i=1}^{n-d} i a_i = n\phi_q(k-2). \tag{1.5}$$

If $\tilde{\mathscr{C}}$ is projective, we have in addition

$$\sum_{i=2}^{n-d} i(i-1)a_i = n(n-1)\phi_q(k-3). \tag{1.6}$$

Let $\mathscr{C}$ be an $[n, k, d]_q$ code, and let P be a point of multiplicity t in $\tilde{\mathscr{C}}$, $t \geq 0$. Fix a hyperplane Π in $PG(k-1, q)$ with $P \notin \Pi$ and define the projection mapping $\varphi_{P,\Pi}$ by

$$\varphi_{P,\Pi}: \begin{cases} PG(k-1, q)\backslash\{P\} \to \Pi \\ Q \to \Pi \cap \langle P, Q \rangle, \end{cases} \tag{1.7}$$

where $\langle P, Q \rangle$ is the line through the points P and Q. (Generally, if $\mathscr{X}$ is a list of flats of $PG(k-1, q)$ we shall denote by $\langle \mathscr{X} \rangle$ the subspace of $PG(k-1, q)$ generated by the flats from $\mathscr{X}$.) We call the mapping defined by (1.7) a projection with respect to P and Π. It can be easily noted that $\varphi_{P,\Pi}$ maps i-flats containing P into $(i-1)$-flats in Π.

For each point $Q \in \Pi$ define

$$\mu(Q) = |\{R \in \tilde{\mathscr{C}} \mid \varphi(R) = Q\}|. \tag{1.8}$$

For every set of points $\mathscr{F} \subset \Pi$ we define

$$\mu(\mathscr{F}) = \sum_{Q \in \mathscr{F}} \mu(Q). \tag{1.9}$$

For each k'-dimensional flat $\mathscr{F}$ in Π with $k' \leq k-2$, $\mu(\mathscr{F}) \leq \gamma_{k'+1} - t$.

Let Π be a plane (2-flat) in $PG(3, q)$ and let l be a line in Π having $P_0, P_1, \ldots, P_q$ as its points. We shall say that l is of type $(\mu(P_0), \mu(P_1), \ldots, \mu(P_q))$ with respect to a given projection.

In what follows we consider 4-dimensional quaternary codes only. As usual, we call the 0-, 1-, and 2-dimensional flats points, lines, and planes, respectively. Given an $[n, 4, d]_4$ code $\mathscr{C}$, we mean by an i-point a point which has multiplicity i in $\tilde{\mathscr{C}}$. Similarly, i-lines (i-planes) will be lines (planes) containing i points from $\tilde{\mathscr{C}}$ (multiplicities counted).

Let q be a prime power. Consider the plane $PG(2, q)$. A κ-set $\mathscr{S}$ of points in $PG(2, q)$ will be called a (κ, ν)-arc, $\nu \geq 2$, if the following conditions are satisfied:

(i) no $\nu + 1$ points from $\mathscr{S}$ are collinear;

(ii) there exist ν collinear points in $\mathscr{S}$.

A (κ, ν)-arc is *complete* if it is not contained in a $(\kappa + 1, \nu)$-arc. Let $\mathscr{S}$ be a (κ, ν)-arc. A line of $PG(2, q)$ is called an i-secant of $\mathscr{S}$ if it has exactly i points in common with $\mathscr{S}$. The number of i-secants of $\mathscr{S}$ will be denoted by τ_i, $i = 0, 1, \dots, \nu$.

The maximum number of points in a (κ, ν)-arc in $PG(2, q)$ is usually denoted by $m(\nu, q)$. An arc with $m(\nu, q)$ points is obviously complete. A $(\kappa, 2)$-arc with $\kappa = m(2, q)$ is called an *oval*. It is well-known that

$$m(2, q) = \begin{cases} q + 2 & \text{for } q \text{ even,} \\ q + 1 & \text{for } q \text{ odd.} \end{cases} \qquad (1.10)$$

Below we summarize some facts about ovals in the projective plane of order 4 (cf. [4]). As already mentioned, $m(2, 4) = 6$. Two different ovals share at most 3 points. Any two ovals are projectively equivalent. Every line intersects an oval in either 2 or 0 points; there are fifteen 2-secants and six 0-secants. We call them secants and external lines, respectively. Each point not on the oval lies on three secants and two external lines.

We have $m(3, 4) = 9$. There exist four projectively nonequivalent complete $(\kappa, 3)$-arcs. One of them contains 7 points and is thus not maximal. A brief description of the three maximal (9, 3)-arcs is given below (cf. [4]).

($\mathscr{A}1$) The set of all points (x_1, x_2, x_3) satisfying $x_1^3 + x_2^3 + x_3^3 = 0$.

($\mathscr{A}2$) The complement of the union of a conic and two of its tangents.

($\mathscr{A}3$) The complement of three non-concurrent lines.

The intersection numbers for these arcs are presented in the table below.

	τ_0	τ_1	τ_2	τ_3
($\mathscr{A}1$)	0	9	0	12
($\mathscr{A}2$)	2	3	6	10
($\mathscr{A}3$)	3	0	9	9

Given a (9, 3)-arc $\mathscr{A}$ and a point P off $\mathscr{A}$ denote by ρ_i, $i = 0, 1, 2, 3$, the number of lines through P intersecting $\mathscr{A}$ in exactly i points. The different possibilities for the numbers ρ_i are given in the following table.

	ρ_0	ρ_1	ρ_2	ρ_3	Number of points of this type
($\mathscr{A}$1)	0	3	0	2	12
($\mathscr{A}$2)	2	0	0	3	1
	1	1	1	2	6
	1	0	3	1	2
	0	2	2	1	3
($\mathscr{A}$3)	2	0	0	3	3
	1	0	3	1	9

LEMMA 1.2. *Let A, B, C, D be four points, no three of them collinear, in $PG(2, q)$ with q even. Then the points $E = \langle A, B\rangle \cap \langle C, D\rangle$, $F = \langle A, C\rangle \cap \langle B, D\rangle$, $G = \langle A, D\rangle \cap \langle B, C\rangle$ are collinear.*

Proof. Without loss of generality take $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$, $D = (1, 1, 1)$ and the rest is a simple check. ■

LEMMA 1.3. *Let Π_0 be a plane in $PG(3, 4)$ and let $\mathscr{O} \subset \Pi_0$ be an oval. Fix an external line to the oval in Π_0, say l, and denote by Π_i, $i = 1, 2, 3, 4$, the remaining planes through l. Let further $\varphi_P = \varphi_{P,\Pi_1}$ be a projection with respect to $P \in PG(3, 4)\backslash(\Pi_0 \cup \Pi_1)$ and Π_1. Then $\varphi_P(\mathscr{O}) = \varphi_Q(\mathscr{O})$ implies $P = Q$.*

Proof. Suppose $\varphi_P(\mathscr{O}) = \varphi_Q(\mathscr{O})$ and $P \neq Q$. Denote by Δ a plane through $\langle P, Q\rangle$ having a nonempty intersection with $\mathscr{O}$, say $\{R, S\} = \Delta \cap \mathscr{O}$. No three of P, Q, R, S are collinear and Lemma 1.2 implies that $T = \langle P, Q\rangle \cap \langle R, S\rangle$, $U = \varphi_P(R) = \varphi_Q(S)$, $V = \varphi_P(S) = \varphi_Q(R)$ are collinear. Therefore $T \in \langle R, S\rangle \subset \Pi_0$, $T \in \langle U, V\rangle \subset \Pi_1$, and $T \in l$. Now without loss of generality we can put

$$\Pi_0 = \{(x_1, x_2, x_3, x_4) | x_i \in GF(4), x_1 = 0\},$$

$$\Pi_1 = \{(x_1, x_2, x_3, x_4) | x_i \in GF(4), x_2 = 0\},$$

$$l = \{(x_1, x_2, x_3, x_4) | x_i \in GF(4), x_1 = 0, x_2 = 0\},$$

$$\mathscr{O} = \{(0, 1, 0, 0), (0, 1, \omega, 0), (0, 1, 0, \omega), (0, 1, \omega, \omega^2), (0, 1, \omega^2, \omega), (0, 1, \omega^2, \omega^2)\},$$

$$P = (1, 1, 0, 0), Q = (1, 1, a, b), a, b \in GF(4), (a, b) \neq (0, 0).$$

We have $\varphi_P(\mathscr{O}) = \{(1, 0, 0, 0), (1, 0, \omega, 0), (1, 0, 0, \omega), (1, 0, \omega, \omega^2), (1, 0, \omega^2, \omega), (1, 0, \omega^2, \omega^2)\}$, and $(a, b) \in \{(\omega, 0), (0, \omega), (\omega, \omega^2), (\omega^2, \omega), (\omega^2, \omega^2)\}$. In no case can we get $\varphi_P(\mathscr{O}) = \varphi_Q(\mathscr{O})$, which completes the proof. ■

2. NONEXISTENCE OF $[51, 4, 37]_4$ CODES

LEMMA 2.1. *Suppose* $\mathscr{C}$ *is a* $[51, 4, 37]_4$ *code. Then*

(i) $\gamma_0 = 1$ (*so the code is projective*), $\gamma_1 = 4$, $\gamma_2 = 14$;

(ii) *a line in a w-plane contains at most* $(w + 5)/4$ *points of* $\tilde{\mathscr{C}}$;

(iii) $a_2 = a_{10} = 0$;

Proof. (i) $\gamma_2 = 14$ is immediate from the code parameters. By Lemma 1.1, $\gamma_1 \le 4$. In fact, $\gamma_1 = 4$ for otherwise $|\tilde{\mathscr{C}}| \le 1 + 2.21 < 51$. Lemma 1.1 now gives $\gamma_0 = 1$.

(ii) This follows immediately from Lemma 1.1.

(iii) Any 2-plane clearly contains a 2-line, giving a contradiction to (ii), and so $a_2 = 0$. Since $m(3, 4) = 9$ (cf. Section 1), any 10-plane contains a 4-line, again contradicting (ii), and so $a_{10} = 0$. ■

LEMMA 2.2. *Let* Π *be a* 14-*plane. Then we have either*

(i) $\tilde{\mathscr{C}}_\Pi = \Pi \setminus \Delta$, *where* Δ *is a complete* (7, 3)-arc, *or*

(ii) $\tilde{\mathscr{C}}_\Pi = \Pi \setminus (l \cup \{P\} \cup \{Q\})$, *where l is a line in* Π, *and P, Q are two different points from* Π *not on l.*

Proof. Suppose Π does not contain a 0- or 1-line. Then $\Pi \setminus \tilde{\mathscr{C}}$ is a (7, 3)-arc. If it is incomplete, i.e., obtained from one of the (9, 3)-arcs by deleting two points, one can easily check from the tables in Section 1 that it contains external lines. In other words Π contains 5-lines of $\tilde{\mathscr{C}}$, which is impossible. If $\Pi \setminus \tilde{\mathscr{C}}$ is a complete (7, 3)-arc we get (i).

Suppose there is a 1-line in Π, say l', and let $P = l' \cap \tilde{\mathscr{C}}$. Each one of the remaining four lines in Π through P must contain a point which is not in $\tilde{\mathscr{C}}$; therefore, there are at least $4 + 4 > 7$ points in $\Pi \setminus \tilde{\mathscr{C}}$, a contradiction.

If Π contains a 0-line we get easily (ii). ■

Remark 2.3. We will refer to a 14-plane given by Lemma 2.2(i) as a 14-plane of type (B1). Such a plane Π has fourteen 4-lines and seven 2-lines ($\tilde{\mathscr{C}}_\Pi$ is the complement of a Fano subplane of Π). We will refer to a 14-plane given by Lemma 2.2(ii) as a 14-plane of type (B2). Note that neither type of 14-plane contains 1-lines and that only 14-planes of type (B2) have 0- or 3-lines.

COROLLARY 2.4. $a_1 = 0$.

Proof. Suppose Π is a 1-plane and let l be a line in Π containing the point from $\tilde{\mathscr{C}}$. Lemma 2.2 implies that a 14-plane cannot contain a 1-line, so we have $|\tilde{\mathscr{C}}| \leq 1 + 4.12 = 49$, which is impossible. ∎

LEMMA 2.5. *For a* $[51, 4, 37]_4$ *code* $\mathscr{C}$, $a_3 = 0$.

Proof. Assume the contrary and let Π_0 be a 3-plane. We are going to show that in such case $PG(3, 4)$ does not contain 6-, 7-, 8-, and 9-planes.

Suppose Π_1 is a 6-plane. Then $l = \Pi_0 \cap \Pi_1$ is a 0-line. Let P_1, P_2 be the two points on l which do not lie on a 2-line in Π_0. The remaining three planes through l, say Π_2, Π_3, Π_4, are 14-planes of type (B2). Denote by R_i, S_i, $i = 2, 3, 4$, the 0-points in $\Pi_i \backslash l$.

Consider a projection φ with respect to P_1 and a plane Π, $P_1 \notin \Pi$. Set $l_i = \varphi(\Pi_i)$, $i = 0, 1, \ldots, 4$. The line l_0 is of type (0, 1, 1, 1, 0) and, since $\tilde{\mathscr{C}}_{\Pi_1}$ is an oval, l_1 is of type (0, 2, 2, 2, 0). Let X_1, X_2, X_3 be the points of l_1 with $\mu(X_i) = 2$. Through each line $\langle P, X_i \rangle$ passes at least one 14-plane for otherwise $|\tilde{\mathscr{C}}| \leq 6 + 4.11 = 50$. Hence there exists, for $i = 1, 2, 3$, a line m_i in Π through X_i such that $\mu(m_i) = 14$. Since 14-planes cannot contain 1-lines, we must have $\mu(m_i \cap l_0) = 0$. Hence m_1, m_2, m_3 are all of type (0, 4, 4, 4, 2), and this in turn implies that l_2, l_3, l_4 are all of type (0, 4, 4, 4, 2). This means that $P_1 \in \langle R_i, S_i \rangle$ for $i = 2, 3, 4$. In the same way we can prove that $P_2 \in \langle R_i, S_i \rangle$, which is impossible.

Now let Π_1 be an 8- or 9-plane. Then $l = \Pi_0 \cap \Pi_1$ is again a 0-line. Let P_1, P_2, Π_2, Π_3, Π_4 be the same as above. At least one of P_1, P_2, say P_1, lies on a 3-secant, say m, to $\tilde{\mathscr{C}}_{\Pi_1}$ (otherwise $(\tilde{\mathscr{C}}_{\Pi_1}) \cup \{P_1, P_2\}$ would be a (10, 3)- or (11, 3)-arc). Consider a projection with respect to P_1 and Π. As before, $l_i = \varphi(\Pi_i)$, $i = 0, \ldots, 4$, $R = \varphi(m)(\mu(R) = 3)$. There exist at least two lines, say $s_1, s_2 \in \Pi$, with $R \in s_1$, $R \in s_2$, $\mu(s_1) = \mu(s_2) = 14$. At least one of them intersects l_0 (which is of type (0, 1, 1, 1, 0)) in a point X with $\mu(X) = 1$, a contradiction to the fact that 14-planes do not contain 1-lines.

Finally, suppose Π_1 is a 7-plane. Once again, $l = \Pi_0 \cap \Pi_1$ is a 0-line and let $P \in l$ be a point lying on a 3-secant to $\tilde{\mathscr{C}}_{\Pi_1}$, say m. Let φ be a projection with respect to P and II, and let $l_i = \varphi(\Pi_i)$, $i = 0, \ldots, 4$, $R = \mu(m)$. Each line $s \in \Pi$ with $R \in s$, $s \neq l_1$, has $\mu(s) = 14$. Therefore, for each $Y \in l_0$ we have $\mu(Y) \neq 1$. This contradicts the fact that l_0 is of type (0, 1, 1, 1, 0) or (0, 2, 1, 0, 0).

It is easily checked that $a_3 > 0$ implies $a_0 = a_4 = a_5 = 0$. Now $154(1.4) - 24(1.5) + (1.6)$ gives

$$-2a_{12} - 2a_{13} = 48,$$

a contradiction. ∎

In order to show that $a_4 = a_5 = 0$ for a $[51, 4, 37]_4$ code, it is necessary first to prove some results about a $[52, 4, 38]_4$ code. Of course, it will eventually follow from our main result that a $[52, 4, 38]_4$ code does not exist, but at this stage we cannot assume this.

LEMMA 2.6. *Suppose $\mathscr{C}$ is a $[52, 4, 38]_4$ code. Then*

(i) $\gamma_0 = 1, \gamma_1 = 4, \gamma_2 = 14$;

(ii) *a line in a w-plane contains at most* $1 + w/4$ *points of* $\tilde{\mathscr{C}}$;

(iii) $a_2 = a_3 = a_7 = a_{10} = a_{11} = 0$;

(iv) $a_0 = 0$;

(v) $a_4 = a_5 = 0$;

(vi) $a_6 = 0$.

Proof. (i) $\gamma_2 = 14$ is immediate from the code parameters. By Lemma 1.1, $\gamma_1 \leq 4$. In fact, $\gamma_1 = 4$ for otherwise $|\tilde{\mathscr{C}}| \leq 1 + 2.21 < 52$. Lemma 1.1 now gives $\gamma_0 = 1$.

(ii) This follows immediately from Lemma 1.1.

(iii) From the values of $m(\nu, 4)$ given in Section 1, it follows that every 2- or 3-plane contains a line with at least two points of $\tilde{\mathscr{C}}$, every 7-plane contains a line with at least three points of $\tilde{\mathscr{C}}$, and every 10- or 11-plane contains a line with at least four points of $\tilde{\mathscr{C}}$. Hence we get a contradiction to (ii) if any of the given a_i's is nonzero.

(iv) Note that $a_0 \geq 1$ implies $a_0 = 1$ and $a_i = 0$ for $i = 1, 2, \ldots, 11$. Now it is easily found that Eqs. (1.4)–(1.6) have the unique solution $a_0 = 1$, $a_{12} = 78$, $a_{13} = -72$, $a_{14} = 78$, which is impossible since a_{13} cannot be negative.

(v) Suppose $a_4 \neq 0$ and Π is a 4-plane. No three of the points in $\tilde{\mathscr{C}}_\Pi$ are collinear; therefore, they define an oval $\mathscr{O}$. Let $Q \in \mathscr{O} \setminus \tilde{\mathscr{C}}$. Let further l be a line through Q, not in Π. Consider the planes Δ_i, $i = 0, 1, \ldots, 4$, containing l. Without loss of generality $\Delta_i \cap \Pi$, $i = 0, 1, 2, 3$, are 1-lines and, as by Lemma 2.2 14-planes do not contain 1-lines we have $|\mathscr{C}_{\Delta_i}| \leq 13$, $i = 0, 1, 2, 3$. This implies

$$|\tilde{\mathscr{C}}| = \sum_{i=0}^{4} |\tilde{\mathscr{C}}_{\Delta_i}| - 4|\tilde{\mathscr{C}}_l| \leq 4.13 + 14 - 4|\tilde{\mathscr{C}}_l|,$$

whence $|\tilde{\mathscr{C}}_l| \leq 14/4$. So, every line through Q has at most three points from $\tilde{\mathscr{C}}$. In fact, an easy counting shows that each line through Q off Π is a 3-

line. Therefore, each plane containing Q has at most thirteen points from $\tilde{\mathscr{C}}$. But now $\tilde{\mathscr{C}} \cup \{Q\}$ gives a $[53, 4, 39]_4$ code, which is a contradiction, as a code with such parameters does not exist [1].

By exactly the same arguments, if Π is a 5-plane, then we may adjoin the sixth point of the oval containing $\tilde{\mathscr{C}}_\Pi$ to $\tilde{\mathscr{C}}$ to get a $[53, 4, 39]_4$ code, which is a contradiction.

(vi) Suppose $a_6 \neq 0$. Equalities (1.4)–(1.6) combined with $a_0 = a_1 = \cdots = a_5 = 0$ and $a_7 = a_{10} = a_{11} = 0$ imply

$$a_{12} + 10a_9 + 15a_8 + 28a_6 = 169. \tag{2.1}$$

Fix a 6-plane Π. For a line l in Π consider the quadruples of nonnegative integers

$$(|\tilde{\mathscr{C}}_{\Pi_1}|, |\tilde{\mathscr{C}}_{\Pi_2}|, |\tilde{\mathscr{C}}_{\Pi_3}|, |\tilde{\mathscr{C}}_{\Pi_4}|), \tag{2.2}$$

where Π_i, $i = 1, 2, 3, 4$, are the planes through l different from Π. If l is a 2-line we have two possibilities for (2.2):

(A) (14, 14, 14, 12)
(B) (14, 14, 13, 13).

If l is a 0-line then (2.2) is one of the following:

(C) (14, 14, 12, 6)
(D) (14, 14, 9, 9)
(E) (14, 13, 13, 6)
(F) (14, 12, 12, 8)
(G) (13, 13, 12, 8)
(H) (13, 12, 12, 9).

As $\tilde{\mathscr{C}}_\Pi$ is an oval there are fifteen 2-lines and six 0-lines in Π. If we assume $a_6 = 1$ the sum (2.1) is maximal if we take the planes through a 2-line to be all of type (A) and the planes through a 0-line to be all of type (D). Hence

$$a_{12} + 10a_9 + 15a_8 + 28a_6 \leq 28 + 15.1 + 6.20 < 169,$$

a contradiction. So, $a_6 \neq 0$ forces $a_6 \geq 2$.

Now let Π_0 and Π_1 be 6-planes. Let $l = \Pi_0 \cap \Pi_1$ (l is obviously a 0-line), and denote by Π_2, Π_3, Π_4 the remaining planes through l. Further write $\tilde{\mathscr{C}}_{\Pi_0} = \{P_i | i = 1, 2, \ldots, 6\}$, $\tilde{\mathscr{C}}_{\Pi_1} = \{Q_j | j = 1, 2, \ldots, 6\}$. Each one of the lines $\langle P_i, Q_j \rangle$, $i, j = 1, 2, \ldots, 6$, must contain a 0-point. On the other hand, a point from $PG(3, 4) \backslash (\Pi_0 \cup \Pi_1)$ lies on at most six such lines.

Suppose there is a point $R \in PG(3, 4) \backslash (\Pi_0 \cup \Pi_1)$ lying on at least four lines from $\{\langle P_i, Q_j \rangle \mid i, j = 1, 2, \ldots, 6\}$, say $R \in \langle P_i, Q_i \rangle$, $i = 1, 2, 3, 4$. Let $\langle R, P_5 \rangle \cap \Pi_1 = Q_5'$, and $\langle R, P_6 \rangle \cap \Pi_1 = Q_6'$. Then $\{Q_1, Q_2, \ldots, Q_6\}$ and $\{Q_1, Q_2, Q_3, Q_4, Q_5', Q_6'\}$ are ovals and we arrive at a contradiction unless $Q_5 = Q_5'$, $Q_6 = Q_6'$. Furthermore, Lemma 1.3 implies that there cannot exist two points in $PG(3, 4) \backslash (\Pi_0 \cup \Pi_1)$ lying on more than 3 lines from $\{\langle P_i, Q_j \rangle | i, j = 1, 2, \ldots, 6\}$. So, if we denote by z the number of 0-points not on Π_0 or Π_1, we get $6 + 3(z - 1) \geq 36$. This implies $z \geq 11$, a contradiction since $z = 8$. ■

LEMMA 2.7. *For a* $[51, 4, 37]_4$ *code* $\mathscr{C}$ *we have* $a_4 = a_5 = 0$.

Proof. Let Π_0 be a 4-plane, and let P, Q be the points on Π_0 for which $(\tilde{\mathscr{C}}_{\Pi_0}) \cup \{P, Q\}$ is an oval. Let l be a 1-line through P and let Π_i, $i = 1, \ldots, 4$, be the other four planes through l. Consider a projection $\varphi = \varphi_{P,\Pi}$, $P \notin \Pi$. Set $l_i = \varphi(\Pi_i)$, $i = 0, \ldots, 4$. The line l_0 is of type (1, 1, 1, 1, 0), $\mu(l_i) \leq 13$, $i = 1, \ldots, 4$.

Assume that for some $X \in \Pi \backslash l_0$, $\mu(X) = 4$. Then there exist at least two lines on Π, says s_1, s_2, through X with $\mu(s_i) = 14$, $i = 1, 2$. For at least one of them, say s_1, we have $\mu(s_1 \cap l_0) = 1$, a contradiction. Therefore, for every $X \in \Pi \backslash l_0$, $\mu(X) \leq 3$. Hence for every line m on Π, $\mu(m) \leq 13$. This means that P does not lie on a 14-plane and $\tilde{\mathscr{C}} \cup \{P\}$ gives a $[52, 4, 38]_4$ code with a 5-plane, a contradiction to Lemma 2.6(v).

Now let Π_0 be a 5-plane and let P be the point of Π_0 such that $\tilde{\mathscr{C}}_{\Pi_0} \cup \{P\}$ is an oval. Any plane, other than Π_0, through P must meet Π_0 in a 1-line and so cannot be a 14-plane. Thus $\tilde{\mathscr{C}} \cup \{P\}$ gives a $[52, 4, 38]_4$ code with a 6-plane, contradicting Lemma 2.6(vi). ■

For future reference let us note that from (1.4)–(1.6) we now have

$$a_{12} + 3a_{11} + 10a_9 + 15a_8 + 21a_7 + 28a_6 + 91a_0 = 187. \qquad (2.3)$$

LEMMA 2.8. *for a* $[51, 4, 37]_4$ *code* $\mathscr{C}$, $a_0 = 0$.

Proof. Suppose $a_0 > 0$. Then $a_0 = 1$ and $a_i = 0$ for $1 \leq i \leq 8$. From (2.3) we have

$$a_{12} + 3a_{11} + 10a_9 = 96 \qquad (2.4)$$

Let Π_0 be the 0-plane. For a line l in Π_0 consider the quadruples

$$(|\tilde{\mathscr{C}}_{\Pi_1}|, |\tilde{\mathscr{C}}_{\Pi_2}|, |\tilde{\mathscr{C}}_{\Pi_3}|, |\tilde{\mathscr{C}}_{\Pi_4}|),$$

where $\Pi_1, \ldots, \Pi_4$ are the planes through l different from Π_0. The possible quadruples are

$$(14, 14, 14, 9), (14, 14, 12, 11), (14, 13, 13, 11),$$
$$(14, 13, 12, 12), (13, 13, 13, 12).$$

Suppose $a_9 = 0$. Then the maximum contribution that the planes through l can make to the left-hand side of (2.4) is 4 (when the quadruple is (14, 14, 12, 11)). Thus the left-hand side of (2.4) is at most $4.21 = 84$, a contradiction. Hence $a_9 > 0$.

Let Π_1 be a 9-plane. The line $l = \Pi_0 \cap \Pi_1$ is a 0-line, and $\tilde{\mathscr{C}}_{\Pi_1}$ is a (9, 3)-arc of type ($\mathscr{A}2$) or ($\mathscr{A}3$). The other three planes through l (we denote them by Π_2, Π_3, Π_4) are 14-planes of type (B2).

Denote by R_i, S_i, $i = 2, 3, 4$, the 0-points on $\Pi_i \backslash l$. Now we consider projections $\varphi_P = \varphi_{P,\Pi}$, $P \notin \Pi$, for different choices of the point $P \in l$. Once again, we set $l_i = \varphi_P(\Pi_i)$.

Firstly, let P lie on three 3-secants and two external lines to $\tilde{\mathscr{C}}_{\Pi_1}$; in other words, let P be a point with $(\rho_0, \rho_1, \rho_2, \rho_3) = (2, 0, 0, 3)$ (see Section 1). Then l_1 is of type (0, 3, 3, 3, 0) and l_2, l_3, l_4 are of type (0, 4, 4, 4, 2) or (0, 4, 4, 3, 3). The set

$$\mathscr{S} = \{X \mid X \in l_2 \cup l_3 \cup l_4, \mu(X) = 4\} \cup \{Y \mid Y \in l_1, \mu(Y) = 3\}$$

is an $(|\mathscr{S}|, 3)$-arc; therefore, $|\mathscr{S}| \leq 9$. This implies that l_2, l_3, l_4 are all of type (0, 4, 4, 3, 3) or, in other words, none of the lines $\langle R_2, S_2\rangle$, $\langle R_3, S_3\rangle$, $\langle R_4, S_4\rangle$ meets P.

Now suppose P lies on one 3-secant, three 2-secants, and one external line to $\tilde{\mathscr{C}}_{\Pi_1}$, i.e., $(\rho_0, \rho_1, \rho_2, \rho_3) = (1, 0, 3, 1)$. Using the same argument about $\mathscr{S}$ we get that not all of l_2, l_3, l_4 are of type (0, 4, 4, 4, 2). Suppose exactly one of l_2, l_3, l_4 is of type (0, 4, 4, 4, 2). Then $\mathscr{S}$ is an (8, 3)-arc which can be extended to a (9, 3)-arc $\mathscr{S}^*$ of type ($\mathscr{A}2$) or ($\mathscr{A}$ 3). Therefore, there exists an external line, say $m \neq l_0$, to $\mathscr{S}$. Then

$$\mu(m) = \sum_{i=0}^{4} \mu(m \cap l_i) = 0 + 2 + 2 + 3 + 3 = 10,$$

a contradiction (Lemma 2.1(iii)). We can conclude that if P has $(\rho_0, \rho_1, \rho_2, \rho_3) = (1, 0, 3, 1)$ then it lies on an even number of lines from $\{\langle R_i, S_i\rangle, i =$

2, 3, 4}. This proves that $\tilde{\mathscr{C}}_{\Pi_1}$ cannot be of type ($\mathscr{A}3$), as in this case all the points on l have $(\rho_0, \rho_1, \rho_2, \rho_3) = (2, 0, 0, 3)$, or $(1, 0, 3, 1)$.

Let now $P \in l$ be a point with $(\rho_0, \rho_1, \rho_2, \rho_3) = (1, 1, 1, 2)$. The argument about $\mathscr{S}$ gives us that at most one of l_2, l_3, l_4 is of type (0, 4, 4, 4, 2). If exactly one of these lines is of type (0, 4, 4, 4, 2) there exists $m \in \Pi$, $m \neq l_0$, which is external to $\mathscr{S}$, with $\mu(m) = 9$. In other words, there exists a 9-plane through P, different from Π_1. Note that we can always choose a point P on l with $(\rho_0, \rho_1, \rho_2, \rho_3) = (1, 1, 1, 2)$ lying on exactly one of $\langle R_2, S_2\rangle$, $\langle R_3, S_3\rangle$, $\langle R_4, S_4\rangle$.

Now let $P' \in l$ be the point with $(\rho_0, \rho_1, \rho_2, \rho_3) = (2, 0, 0, 3)$ and $P'' \in l$ be a point with $(\rho_0, \rho_1, \rho_2, \rho_3) = (1, 1, 1, 2)$ lying on exactly one of the lines $\langle R_i, S_i\rangle$, $i = 2, 3, 4$. There exists a 9-plane $\Delta_1 \neq \Pi_1$ through P''. Note that $\tilde{\mathscr{C}}_{\Pi_1}$ and $\tilde{\mathscr{C}}_{\Delta_1}$ are (9, 3)-arcs of type ($\mathscr{A}2$). Denote by s (resp. t) the 0-line in Π_1 (resp. Δ_1), which is not in Π_0. Obviously, $P' \in s$, $P' \notin t$. Write $R = t \cap \Pi_0$.

Suppose there exists a plane Γ containing both s and t. Then Γ contains three non-concurrent 0-lines (s, t and $\langle P', R\rangle$) and must be a 9-plane. $\tilde{\mathscr{C}}_\Gamma$ is a (9, 3)-arc of type ($\mathscr{A}3$), which was shown to be impossible. Therefore, s and t have to be skew lines.

To complete the proof we are going to show that there cannot exist two skew 0-lines off Π_0. Denote by $\mathscr{K}$ the set of all 0-points in $PG(3, 4)$. Let further $\mathscr{K}_0 = \mathscr{K} \setminus (\Pi_0 \cup s \cup t)$. We have $|\mathscr{K}_0| = 5$. For a plane Γ with $\Gamma \supset s$, $R \notin \Gamma$, $\mathscr{K}_0 \cap \Gamma \neq \emptyset$, we have $|\tilde{\mathscr{C}}_\Gamma| \leq 10$. Therefore, Γ is a 9-plane, i.e., $|\Gamma \cap \mathscr{K}_0| = 2$. Hence $\langle s, P', R\rangle$ contains only one point from $\mathscr{K}_0$ and is thus an 11-plane. Similarly, $\langle t, P', R\rangle$ is an 11-plane. Counting the number of points on the planes through $\langle P', R\rangle$ we get

$$|\tilde{\mathscr{C}}| = \sum_{\langle P',R\rangle \subset \Gamma} |\tilde{\mathscr{C}}_\Gamma| \leq 0 + 2.11 + 2.14 = 50,$$

a contradiction. ∎

LEMMA 2.9. *Let $\varphi_{P,\Pi}$ be a projection and suppose A_0, A_1, A_2 are points in Π with $\mu(A_i) = 0$. Then A_0, A_1, A_2 are not collinear.*

Proof. Suppose A_0, A_1, A_2 lie on a line l and let $X \in l$, $X \neq A_i$, $i = 0, 1, 2$. Now $\mu(X) \neq 4$ because a plane with fewer than 11 points cannot contain a 4-line (Lemma 2.1(ii)). Furthermore, $\mu(X) \neq 3$ because a plane with fewer than 7 points cannot contain a 3-line. Hence $\mu(l) \leq 4$, which is impossible since we have shown that $a_i = 0$ for $i \leq 4$. ∎

LEMMA 2.10. *Suppose $\mathscr{C}$ is a $[51, 4, 37]_4$ code. Then $a_7 = a_8 = 0$.*

Proof. Let Π_0 be a 7- or 8-plane and let $l \in \Pi_0$ be a 3-line. Denote by

Π_i, $i = 1, 2, 3, 4$, the remaining planes through l. Without loss of generality, Π_1, Π_2, Π_3 are 14-planes of types (B2). Consider a projection $\varphi = \varphi_{P,\Pi}$, $P \notin \Pi$, where P is a 0-point of l. Let $l_i = \varphi(\Pi_i)$, $i = 0, ..., 4$. The point P can be so chosen that at least two of the lines l_1, l_2, l_3, say l_1 and l_2, are of type (3, 4, 4, 3, 0) (consider where the 0-lines of Π_1, Π_2, Π_3 meet l).

Denote by A_i, $i = 1, 2$, the points with $A_i \in l_i$, $\mu(A_i) = 0$. Let further $m_0 = \langle A_1, A_2 \rangle$ and $A_0 = m_0 \cap l_0$. Denote by m_i, $i = 1, 2, 3$, the lines in Π through A_0, different from m_0 and l_0.

We have $\mu(A_0) = 1, 2$, or 3 ($\mu(A_0) = 0$ is impossible by Lemma 2.9, $\mu(A_0) = 4$ is impossible by Lemma 2.1(ii)). It is easily seen that

$$|\tilde{\mathscr{C}}| = 51 = \sum_{i=0}^{3} \mu(m_i) + \mu(l_0) - 4\mu(A_0). \tag{2.5}$$

Suppose $\mu(A_0) = 1$. Then $\mu(m_i) \le 13$, $i = 1, 2, 3$, $\mu(m_0) \le 7$ and (2.5) becomes $51 \le 3.13 + 7 + 8 - 4.1 = 50$, a contradiction. Now let $\mu(A_0) = 2$. We have $\mu(m_i) \le 14$, $i = 1, 2, 3$, $\mu(m_0) \le 8$ and from (2.5), $51 \le 3.14 + 8 + 8 - 4.2 = 50$, a contradiction. At last let $\mu(A_0) = 3$. This time $\mu(m_0) \le 11$ and (2.5) gives again a contradiction $51 \le 3.14 + 11 + 8 - 4.3 = 49$. ■

LEMMA 2.11. *For a* $[51, 4, 37]_4$ *code* $\mathscr{C}$, $a_6 = 0$.

Proof. First of all, let us note that if $a_6 \ge 2$ we obtain a contradiction as in Lemma 2.6(vi). Now suppose that $a_6 = 1$ and let Π_0 be the 6-plane. From (1.4)–(1.6) we get that in such case $a_9 > 0$ for otherwise 154(1.4) − 24(1.5) + (1.6) gives $2a_{12} + 2a_{13} = -96$. Let Π_1 be a 9-plane. The line $l = \Pi_0 \cap \Pi_1$ is a 0-line and $\tilde{\mathscr{C}}_{\Pi_1}$ is a (9, 3)-arc of type ($\mathscr{A}$2) or ($\mathscr{A}$3). Let $P \in l$ be a point lying on three 3-secants and two external lines to $\tilde{\mathscr{C}}_{\Pi_1}$. Consider the projection $\varphi = \varphi_{P,\Pi}$, $P \notin \Pi$. Set $l_i = \varphi(\Pi_i)$, $i = 0, 1$. The line l_0 is of type (0, 2, 2, 2, 0) and l_1 is of type (0, 3, 3, 3, 0).

Fix $A \in l_0$ with $\mu(A) = 2$. Let l' be the line in Π_0 with $\varphi(l') = A$. Let Δ be a 14-plane containing l' (such a plane must exist, for otherwise $|\tilde{\mathscr{C}}| \le 6 + 4.11 = 50$). Since Δ meets Π_1 in a 0- or 3-line, Δ is of type (B2). Let m be the 0-line of Δ and let Δ_i, $i = 1, 2, 3, 4$, be the other planes through m. Then $\sum_{i=1}^{4} |\tilde{\mathscr{C}}_{\Delta_i}| = 37$, where each of the numbers $|\tilde{\mathscr{C}}_{\Delta_i}|$ is 9, 11, 12, 13 or 14 (note that Π_0 is the only i-plane with $i < 9$). Clearly, we cannot find four such numbers which sum to 37. ■

THEOREM 2.12. *There is no* $[51, 4, 37]_4$ *code.*

Proof. Suppose $\mathscr{C}$ is a $[51, 4, 37]_4$ code and Π_0 is a 14-plane of type (B2). Denote by l the 0-line in Π_0, and by Π_i, $i = 1, 2, 3, 4$, the remaining planes through l. Then $51 = \sum_{i=0}^{4} |\tilde{\mathscr{C}}_{\Pi_i}|$, i.e., $\sum_{i=1}^{4} |\tilde{\mathscr{C}}_{\Pi_i}| = 37$, where each of the numbers $|\tilde{\mathscr{C}}_{\Pi_i}|$ is 9, 11, 12, 13, or 14. Again, we cannot find four such numbers which sum to 37.

Now let Π_0 be a 14-plane of type (B1). For different choices of the line $l \in \Pi_0$ consider the quadruples of nonnegative integers

$$(|\tilde{\mathscr{C}}_{\Pi_1}|, |\tilde{\mathscr{C}}_{\Pi_2}|, |\tilde{\mathscr{C}}_{\Pi_3}|, |\tilde{\mathscr{C}}_{\Pi_4}|),$$

where Π_i, $i = 1, 2, 3, 4$, are the other four planes through l. If l is a 4-line then the quadruple is one of

$$\begin{array}{ll} \text{(A)} & (14, 14, 14, 11) \\ \text{(B)} & (14, 14, 13, 12) \\ \text{(C)} & (14, 13, 13, 13). \end{array}$$

If l is a 2-line then the quadruple is one of

$$\begin{array}{lr} \text{(D)} & (14, 13, 9, 9) \\ \text{(E)} & (14, 11, 11, 9) \\ \text{(F)} & (13, 12, 11, 9) \\ \text{(G)} & (12, 12, 12, 9) \\ \text{(H)} & (12, 11, 11, 11). \end{array}$$

From (2.3) we have

$$a_{12} + 3a_{11} + 10a_9 = 187. \tag{2.6}$$

There are fourteen 4-lines and seven 2-lines in Π_0, so the left-hand side of (2.6) is maximal if we take the planes through the 4-lines to be all of type (A) and the planes through the 2-lines to be all of type (D). Hence,

$$187 = a_{12} + 3a_{11} + 10a_9 \leq 14.3 + 7.20 = 182,$$

a contradiction. ■

Remark 2.13. The results of this paper also make a contribution to the theory of so-called minihypers. An $\{f, m; r, q\}$ minihyper is defined to be a set of f points in $PG(r, q)$ which meets every hyperplane in at least m points. Minihypers have been studied extensively in connection with the problem of finding and classifying codes meeting the Griesmer bound; see [2] for a recent survey. If a $[51, 4, 37]_4$ code is viewed as a 51-set in $PG(3,4)$ which meets every plane in at most 14 points, then its complement in $PG(3, 4)$ is a {34, 7; 3, 4} minihyper. It is thus proved in Theorem 2.12 that {34, 7; 3, 4} minihypers do not exist.

References

1. P. Greenough and R. Hill, Optimal linear codes over GF (4), *Discrete Math.* **125** (1994), 187–199.
2. N. Hamada, A survey of recent work on characterization of minihypers in $PG(t, q)$ and nonbinary linear codes meeting the Griesmer bound, *J. Combin. Inform. System Sci 18* (1993), 161–191.
3. R. Hill and I. Landgev, On the nonexistence of some quaternary codes, submitted for publication.
4. J. W. P. Hirschfeld, "Projective Geometries over Finite Fields," Clarendon Press, Oxford, UK, 1979.