ON MATRIX QUANTUM GROUPS OF TYPE A_n

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Abstract. To a Hecke symmetry R there associate a matrix bialgebra E_R and a matrix Hopf algebra H_R, which are called function rings on the matrix quantum semi-group and matrix quantum groups associated to R. We show that for an even Hecke symmetry, the rational representations of the corresponding quantum group are absolutely reducible and compute the integral on the function ring of the quantum group, i.e, on H_R. Further, we show that the fusion coefficients of simple representations depend only on the rank of the symmetry, and give the explicit formula for the rank or 8-dim of H_R-comodules. In the general case, we show that the quantum semi-group is "Zariski" dense in the quantum group. This enables us to study the semi-simplicity of the associated quantum group in some case.

1. Introduction

Matrix quantum groups of type A_n generalize the standard deformations of the matrix group $\mathrm{GL}(n)$ introduced by Faddeev, Reshetikhin, Takhtajan [4] as well as multi-parameter deformations of the super-group $\mathrm{GL}(m|n)$ introduced by Manin [23]. In fact, to every Hecke symmetry R there associate a bialgebra E_R and a Hopf algebra H_R, that are considered as the ring of regular functions on the corresponding matrix quantum semi-group and group respectively. In case when R is the Drinfel'd-Jimbo matrix of type A_n or its super analog, this Hopf algebra reduces to the mentioned above deformations. These quantum groups were firstly studied by Gurevich [6], who generalized Lyubashenko's results on vector symmetry to the case of Hecke symmetry.

Many interesting properties of quantum matrix (semi-) groups were found [7, 20, 10, 24, 8]. These results show on one hand the similarity in the theory of quantum matrix groups of type A_n and the classical groups $\mathrm{GL}(n)$ and $\mathrm{GL}(m|n)$, and on the other hand the relations between quantum matrix groups of type A_n and the Hecke algebras.

In this work we study rational representations of matrix quantum groups of type A_n, i.e., comodules of the corresponding Hopf algebras. In case R is an even Hecke symmetry of rank r, we show that the category of comodules of R is equivalent to the category of representations of the standard quantum groups $\mathrm{GL}_q(r)$ introduced by Faddeev, Reshetikhin and Takhtajan [4]. We show that this category is a ribbon category and explicitly compute the rank or 8-dim as well as the ribbon-dimension of simple comodules. Further we give the explicit formula for the integral on H_R. This formula seems to be new even in the case of $\mathrm{GL}(r)$.

According to our results, one can conclude that, over a field of characteristic zero, to each non-zero and not root of unity q there exists *one* quantum general linear groups $\mathbf{GL}_q(n)$, which is to be understood as an abstract abelian braided semi-simple category. The fusion coefficients of $\mathbf{GL}_q(n)$ (to be understood as a category) do depend only on n but not on q. Each even Hecke symmetry of rank n provides a functor from $\mathbf{GL}_q(n)$ to the category of vector spaces which is to

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be understood as a realization of $\mathbf{GL}_q(n)$, and such a realizion exists for every n and q (provided that q possesses a square root). An important invariant of $\mathbf{GL}_q(n)$ is the 8-dim which shows that $\mathbf{GL}_q(n)$'s for different q are not equivalent. The formula for 8-dim on simple comodules is explicitly computed, which is quite analog to Weyl's formula for rational simple $\mathrm{GL}(r)$-modules.

For an arbitrary Hecke symmetry, we show that the matrix quantum groups is "Zariski" dense in the corresponding quantum semi-groups, that is, in dual, the canonical homomorphism $E_R \longrightarrow H_R$ is injective. This fact enables us to study the semi-simpicity of H_R in some special case. Our technique bases mainly on the theory of Hecke algebras.

The work is organized as follows. In Section 2 we briefly recall some definitions of the bialgebra E_R and its Hopf envelope H_R. In Section 3 we study the case where R is an even Hecke symmetry. Section 4 is devoted to the general case. In the Appendix we briefly recall some results on Hecke algebra from [1, 2] and prove some lemmas needed in our context.

Let us fix a filed k of characteristic zero. All objects of the paper are defined over k. Throughout the paper we shall be dealt with an element q from $k^\times$ which we will assume *not to be a root of unity, but may be the unity itself.*

2. The Matrix Quantum Groups and Quantum Semi-groups

Let V be a finite dimensional vector space over k and $R : V \otimes V \longrightarrow V \otimes V$ be an invertible k-linear operator. Fix a basis $x_1, x_2, \ldots, x_d$ of V and let $\xi^1, \xi^2, \ldots, \xi^d$ be its dual basis in the dual vector space V^* to V, $< \xi^i | x_j >= \delta^i_j$. Denote $z^i_j := \xi^i \otimes x_j$, then $z^i_j, 1 \le i, j \le d$ form a k-linear basis of $V^* \otimes V$. Let R^{kl}_{ij} be the matrix of R with respect to the basis $x_i \otimes x_j$, $x_i \otimes x_j R = x_k \otimes x_l R^{kl}_{ij}$. The algebra E_R is defined to be the factor algebra of the tensor algebra on $V^* \otimes V$ by the relation:

$$R^{mn}_{ij} z^k_m z^l_n = z^p_i z^q_j R^{kl}_{pq}. \tag{1}$$

Remark. The definition of E_R is, in fact, basis free.

E_R is a bialgebra, the coproduct and counit are given by $\Delta(z^i_j) = z^i_k \otimes z^k_j$ and $\varepsilon(z^i_j) = \delta^i_j$ respectively (we shall frequently use the convention of summing up by the indices that appear both in lower and upper places). We shall be interested in right E_R-comodules, the coaction of E_R will be denoted by δ. E_R is called a matrix quantum semi-group iff R satisfies the Yang-Baxter equation:

$$(R \otimes V)(V \otimes R)(R \otimes V) = (V \otimes R)(R \otimes V)(V \otimes R),$$

and matrix quantum semi-group of type A_n iff R also satisfies the Hecke equation:

$$(R+1)(R-q) = 0,$$

for some $q \in k^\times$ ($k^\times := k \setminus \{0\}$). In this paper we shall be interested in the latter case, i.e., the operator R is assumed to satisfy both equations. We call such an operator Hecke operator. We define the following algebras. Λ_R and S_R are factor algebras of the tensor algebra upon V by the following relations:

$$\begin{aligned} -x_i \otimes x_j &= x_k \otimes x_l R^{kl}_{ij}, \\ q x_i \otimes x_j &= x_k \otimes x_l R^{kl}_{ij}, \end{aligned} \tag{2}$$

respectively. These algebras play the analogous role of the antisymmetric and symmetric tensor algebras on V.

The most well known example of R is the Drinfel'd-Jimbo matrix $R^{q,d}$ given by:

$$R^d := q \sum_{i,j=1}^{d} e^{ij}_{ij} - (q+1) \sum_{i,j=1}^{d} \Big(\frac{q^{\varepsilon(ij)}}{1 + q^{\varepsilon(ij)}} e^{ij}_{ij} - \frac{q^{\varepsilon(ij)/2}}{1 + q^{\varepsilon(ij)}} e^{ji}_{ij} \Big), \tag{3}$$

where $\varepsilon(ij) := \mathrm{sign}(i-j)$. R^d is an even Hecke symmetry of rank $d = \dim(V)$. The associated Hopf algebra H_R (see below) is the general linear quantum group $\mathrm{GL}_q(d)$ [4]. Analogously, the general linear quantum supergroup is given by the following matrix:

$$R^{m|n} = q\sum_{i,j=1}^{m+n} e_{ij}^{ij} - (q+1)\sum_{i,j=1}^{m+n}\Big(\frac{q^{\varepsilon(ij)}}{1+q^{\varepsilon(ij)}}e_{ij}^{ij} - (-1)^{\hat{i}\hat{j}}\frac{q^{\varepsilon(ij)/2}}{1+q^{\varepsilon(ij)}}e_{ij}^{ji}\Big), \tag{4}$$

where $\hat{i}$ is the parity of the (homogeneous) basis element x_i. $R^{m|n}$ is a non-even Hecke symmetry if $m, n \neq 0$. The associated Hopf algebra H_R is the general linear quantum supergroups $\mathrm{GL}_q(m|n)$ [23, 20].

The matrix quantum group associated to R is defined to be the Hopf envelope of E_R. By definition, the Hopf envelope of a bialgebra E is a pair (H, i) consisting of a Hopf algebra H and a bialgebra homomorphism $i : E \longrightarrow H$, satisfying the following universal property:

> for any Hopf algebra F and a bialgebra homomorphism $e : E \longrightarrow F$, there exists uniquely a Hopf algebra homomorphism $h : H \longrightarrow F$, such that $e = h \circ i$.

This definition is dual to the definition of the maximal subgroup of a semi-group. Because of that we use this definition as the standard way of finding a quantum group from a given quantum semi-group. The Hopf envelope always exists (cf. [22, 27]).

Let H_R be the Hopf envelope of E_R, then H_R is in general infinitely generated as an algebra. However, if R is a closed Yang-Baxter operator, then H_R will be, in fact, finitely generated. This fact was first proved in [4] with slightly stronger assumption. Since we are most interested in this case, below we will briefly recall the construction of H_R in this case.

Let $R : V \otimes V \longrightarrow V \otimes V$ be a Yang-Baxter operator. Then R induces a braiding in the monoidal category generated by V and R, that is the category $\mathcal{V}$, objects of which are $V^{\otimes n}, n = 0, 1, 2, \ldots$, morphisms are obtained from R and the identity morphisms by composing and taking the tensor product. In particular, the braiding on $V \otimes V$ is R [14]. E_R can be considered as the Coend of the forgetful functor from $\mathcal{V}$ to the category of vector spaces [25]. We want to extend $\mathcal{V}$ to a braided rigid category. The problem is to extend R to a braiding on $V \otimes V^*$ and $V^* \otimes V$, here V^* is understood to be the left dual to V. The candidates of these operators can be the inverse of the half dual $R^\flat$ of R and $(R^{-1})^\flat$ [18]. Let $\mathrm{ev}_V : V^* \otimes V \longrightarrow k$ be the evaluation map, $\mathrm{ev}_V(\xi^i \otimes x_j) = \delta^i_j$, and $\mathrm{db}_V : k \longrightarrow V \otimes V^*$ be the dual basis map, $\mathrm{db}_V(1) = x_i \otimes \xi^i$, then the half dual to R, $R^\flat : V^* \otimes V \longrightarrow V \otimes V^*$ is defined as follows:

$$R^\flat := (\mathrm{ev}_V \otimes V \otimes V)(V \otimes R \otimes V)(V \otimes V \otimes \mathrm{db}_V).$$

In the basis $x_i \otimes x_j$ $R^\flat$ can be given by

$$\xi^i \otimes x_j R^\flat = x_l \otimes \xi^k R^{il}_{jk}.$$

Thus the operator twisting V and V^* can be given by:

$$x_l \otimes \xi^k R_{V,V^*} = \xi^i \otimes x_j P^{kj}_{li},$$

with P satisfying $R^{il}_{jk}P^{km}_{ln} = \delta^i_n\delta^m_j$ (and hence $P^{il}_{jk}R^{km}_{ln} = \delta^i_n\delta^m_j$). It was shown by Lyubashenko that if $R^\flat$ is invertible then so is $(R^{-1})^\flat$ [19], and the operator twisting V^* and V is given by:

$$\xi^i \otimes x_j R_{V^*,V} = x_l \otimes \xi^k {R^{-1}}^{il}_{jk}.$$

A Yang-Baxter operator R is said to be closed iff the operator $R^\flat$ is invertible (and hence so is $(R^{-1})^\flat$). In this case the Hopf envelope H_R can be characterized as the factor algebra of the tensor algebra on the vector space, generated by $z_i^j, t_i^j, 1 \leq i, j \leq d$, by the following relations (cf. [4, 25]):

$$R_{ij}^{mn} z_m^k z_n^l = z_i^p z_j^q R_{pq}^{kl}, \tag{5}$$
$$t_j^i z_k^j = \delta_k^i, \tag{6}$$
$$z_j^i t_k^j = \delta_k^i. \tag{7}$$

The coproduct is: $\Delta(z_i^j) = z_k^j \otimes z_i^k, \Delta(t_i^j) = t_i^k \otimes t_k^j$, $S(x_j^i) = t_j^i$. To show that H_R is a Hopf algebra it is sufficient to find the antipode on t_j^i. Multiplying (5) with t_k^s from the left and t_r^j from the right and using (6), (7), we have

$$R_{ij}^{mn} z_n^l t_q^j = t_k^m z_i^p R_{pq}^{kl} \tag{8}$$

or equivalently

$$P_{lk}^{qp} z_n^l t_q^j = t_k^m z_i^p P_{nm}^{ji}. \tag{9}$$

Setting $n = j$ in (9) and summing up by this index we get $B_k^p = t_k^m z_i^p B_m^i$, or

$$t_k^m B_m^i z_i^p {B^{-1}}_p^j = \delta_k^j, \tag{10}$$

where $B_m^i := P_{lm}^{li}$. Analogously, setting $k = p$ in (9) and summing up by this index, we get:

$${C^{-1}}_k^n z_n^l C_p^q t_q^j = \delta_k^j, \tag{11}$$

where $C_j^i := P_{jl}^{il}$. (10) and (11) imply that

$$S(t_k^j) = B_k^i z_i^p {B^{-1}}_p^j = {C^{-1}}_k^l z_n^l C_l^j. \tag{12}$$

Thus we showed that H_R is a Hopf algebra, hence the Hopf envelope of E_R.

Equation (12) shows that V being a simple comodule on E_R may turn to be reducible being comodule on H_R. In [4], there were introduced a condition ([4], Eq. (20)), which is equivalent to the following:

$$BC = \text{const} \cdot I,$$

that makes the second equation in (12) trivial. It is easy to show (see Section 4) that this equation will be satisfied if R is a Hecke operator. In this case

$$BC = (q^{-1} - (1 - q^{-1}) \operatorname{Tr}(C))I.$$

In Section 4 we will show that if R is a *Hecke symmetry*, i.e., R is a closed Hecke operator, then simple E_R-comodules remain simple being considered as H_R-comodules.

3. Matrix Quantum Groups Associated with Even Hecke Symmetries

If R is an even Hecke symmetry then there exists a one dimension E_R-comodule which yields a group-like element, called quantum determinant, in E_R. The localization of E_R with respect to the quantum determinant is H_R. We study in this section comodules of H_R in this case.

3.1. **Simple Comodules of E_R and H_R.** By its definition, E_R is a graded algebra. Its n$^{\text{th}}$ component E_n is the factor space of $(V^*\otimes V)^{\otimes n}$. E_n is a coalgebra and every E_R-comodule decomposes into direct sum of E_n-comodules [5]. E_n^*, the dual vector space to E_n, is an algebra and acts on every (right) E_n-comodules from the left. In fact, we have an equivalence of the two categories: ${}_{E_n^*}\mathcal{M}\cong\mathcal{M}^{E_n}$.

The Hecke algebra $\mathcal{H}_n$ (see Appendix) acts on $V^{\otimes n}$ from the right via R, for $x\in V^{\otimes n}$,

$$xT_i := xR_i,\ \text{where}\ R_i := \mathrm{id}_{V^{\otimes i-1}}\otimes R\otimes \mathrm{id}_{V^{\otimes n-i-1}}, 1\le i\le n-1.$$

Thus, we have an algebra homomorphism

$$\rho_n : \mathcal{H}_n \longrightarrow \mathrm{End}(V^{\otimes n}), \rho_n(T_i)=R_i. \tag{13}$$

Let denote by $\overline{\mathcal{H}_n}$ the image of $\mathcal{H}_n$ under ρ_n. It was shown in [10] that the actions of $\mathcal{H}_n$ and E_n^* on $V^{\otimes n}$ commute and the following relations hold:

$$E_n^*\cong \mathrm{End}_{\mathcal{H}_n}(V^{\otimes n})\equiv \mathrm{End}_{\overline{\mathcal{H}_n}}(V^{\otimes n}), \tag{14}$$

$$\overline{\mathcal{H}_n}\cong \mathrm{End}_{E_n^*}(V^{\otimes n}). \tag{15}$$

$\mathcal{H}_n$ is semi-simple, in fact, it is direct product of matrix ring over k [1, 2], hence E_n^* and $\overline{\mathcal{H}_n}$ are semi-simple. Let $\mathcal{I} := \mathrm{Ker}(\rho_n)$, then $\overline{\mathcal{H}_n}$ is isomorphic to the two-sided ideal of $\mathcal{H}_n$, $\{h\in\mathcal{H}_n | h\mathcal{I}=0\}$ and $\mathcal{H}_n$ decomposes into a direct product of algebras:

$$\mathcal{H}_n\cong\overline{\mathcal{H}_n}\times\mathcal{I}.$$

Thus, any primitive idempotent of $\mathcal{H}_n$ belong either to $\overline{\mathcal{H}_n}$ or $\mathcal{I}$. By virtue of (14), every simple module of E_n^* is a submodule of $V^{\otimes n}$ considered as an E_n^*-module. Let L be a simple E_n^* module and $i : L\longrightarrow V^{\otimes n}$ be the isomorphism identifying L with a submodule of $V^{\otimes n}$, then i splits. Let $p : V^{\otimes n}\longrightarrow L$ be an E_n^*-comodule homomorphism, such that $pi=\mathrm{id}_L$. Under the isomorphism in (15), ip corresponds to a primitive idempotent in $\overline{\mathcal{H}_n}$. Conversely, let l be a primitive idempotent in $\overline{\mathcal{H}_n}$, then $\mathrm{Im}(l)$ is an E_n^*-module, for if $h\in\overline{\mathcal{H}_n}, x\in V^{\otimes n}$, $h((x)l)=(h(x))l$. Moreover, this E_n^*-module is simple, by virtue of (15). Thus, a primitive idempotent in $\overline{\mathcal{H}_n}$ yields a simple module of E_n^*.

Let $\mathbf{1}=\sum e_i$ be the decomposition of the unit in $\overline{\mathcal{H}_n}$ into central minimal idempotents, then, as an $\overline{\mathcal{H}_n}$-module, $V^{\otimes n}$ is decomposed into:

$$V^{\otimes n}\cong\bigoplus_i \mathrm{Im}(e_i).$$

Therefore E_n^* decomposes into the direct product of its minimal two-sided ideals, which are themselves algebras:

$$E_n^*\cong\prod_i \mathrm{End}_{\overline{\mathcal{H}_n}}\mathrm{Im}(e_i).$$

Hence, a simple E_n^*-module is the simple module of some $E_{n,i}^* := \mathrm{End}_{\overline{\mathcal{H}_n}}\mathrm{Im}(e_i)$, for some uniquely defined i. Therefore, two primitive idempotents in $\overline{\mathcal{H}_n}$ define isomorphic simple E_n^*-module iff they belong to the same minimal two-sided ideal of $\overline{\mathcal{H}_n}$. Thus we proved the following

Lemma 3.1. *Every simple E_n-comodule has the form*

$$L\cong \mathrm{Im}(\rho(l)),$$

where l is a primitive idempotent of $\mathcal{H}_n$. Conversely, every primitive idempotent l of $\mathcal{H}_n$ induces a simple comodule of E_n iff $\rho(l)\neq 0$. Further, the modules defined by l and l' are non-isomorphic iff l and l' belong to different minimal two-sided ideal.

Remark Since $\mathcal{H}_n$ is the direct product of matrix ring over k, so is E_n^*. Therefore Schur's lemma holds for E_R – automorphism ring of a simple E_R-comodule is k.

A Hecke symmetry R is called *even Hecke symmetry* if R the algebra Λ_R defined in (2) is finite dimensional. It was shown by D. Gurevich that in this case E_R owns a group-like element, called quantum determinant, and the localization of E_R with respect to this element is a Hopf algebra. We want to show here that in this case the canonical homomorphism $i : E_R \longrightarrow H_R$ is injective. First, we need the following

Proposition 3.2. *Let R be a Hecke operator. Let r be the largest number, such that $\Lambda_R^r \neq 0$. Assume that $r < \infty$, then Λ_R^r is a simple E_R-comodule and for every simple E_R-comodule N, $\Lambda_R^r \otimes N$ is simple too.*

Proof. It is known (and easy to prove) (cf. [7]) that:

$$\Lambda_R^r = \mathrm{Im}(\rho(y_r)),$$

where $y_r := \frac{1}{[r]_{1/q}} \sum_{w \in \mathfrak{S}_r} (-q)^{-l(w)} T_w$ (see Appendix). Thus, Λ_R^r is a simple E_R-comodule, since y_r is a primitive idempotent. By the assumption of the Proposition we have:

$$\rho(y_r) \neq 0 \text{ and } \rho(y_s) = 0, \forall s \geq r+1.$$

Let N be a simple E_R-comodule of homogeneous degree $m \geq 1$. By Lemma 3.1 there exists a primitive idempotent θ in $\mathcal{H}_m$, such that $N \cong \mathrm{Im}(\rho(\theta))$. $\Lambda_R^r \otimes N$ is a homogeneous comodule of degree $r+m$. Embedding $\mathcal{H}_r$ and $\mathcal{H}_m$ into $\mathcal{H}_{r+m}$ as follows:

$$\mathcal{H}_r \ni T_i \longmapsto T_i \in \mathcal{H}_{r+m}, 1 \leq i \leq r-1$$
$$\mathcal{H}_m \ni T_j \longmapsto T_{r+j} \in \mathcal{H}_{r+m}, 1 \leq j \leq m-1,$$

we have $\Lambda_R^r \otimes N \cong \mathrm{Im}\rho_{r+m}(y_r\theta)$. Thus, it is sufficient to show that $\rho_{r+m}(y_r\theta)$ is a primitive idempotent in $\overline{\mathcal{H}_{r+m}}$. This is precisely the assertion of Lemma A.4.∎

The number r in Proposition is called the rank of R. A direct consequence of the Proposition is that $\Lambda_R^r \otimes \Lambda_R^r$ is a simple comodule. Hence, the braiding on $\Lambda_R^r \otimes \Lambda_R^r$ should be scalar. Since the identity operator is not closed, except when $\dim_k \Lambda_R^r = 1$, R is not closed if $\dim_k \Lambda_R^r \geq 2$ (cf. [7]).

Now assume that R is an even Hecke symmetry, then $\dim_k \Lambda_R^r = 1$, and hence induces the quantum determinant denoted by D in E_R. It is shown in [7], that $E_R[D^{-1}]$ is a Hopf algebra, hence coincides with H_R (cf. [11]). Since H_R is a Hopf algebra, every finite dimensional H_R-comodule possesses the dual. Let Λ_R^{r*} denote the left dual H_R-comodule to Λ_R^r then

$$\Lambda_R^r \otimes \Lambda_R^{r*} \cong \Lambda_R^{r*} \otimes \Lambda_R^r \cong I$$

as H_R-comodules, by definition of a dual comodule and since the antipode sends a group-like element to its inverse.

Theorem 3.3. *Let R be a Hecke symmetry. The the following assertions hold:*
- (i) *The canonical homomorphism $i : E_R \longrightarrow H_R$ is injective,*
- (ii) *H_R is cosemi-simple,*
- (iii) *Simple comodules of H_R have the form $N \otimes {\Lambda_R^r}^{*\otimes n}$, where N is a simple E_R-comodule, $n \geq 0$.*

Proof. According to Proposition 3.2, $N \otimes \Lambda_R^r$ is simple whenever N is. Hence, D is not a zero-divisor in E_R. Thus, the localizing homomorphism $E_R \longrightarrow E_R[D^{-1}] = H_R$ is injective, (i) is proved.

Let M be a finite dimensional comodule on $H_R \cong E_R[D^{-1}]$, and $x_i, i = 1, 2, \ldots, m$ be a k-basis of M, then there exist elements $a^i_j, 1 \le i, j \le m$ in H_R, such that $\delta(x_i) = x_j a^j_i$. Let n be such an integer, that $a^i_j D^n \in E_R, \forall i, j$. Then $M \otimes {\Lambda^r_R}^{\otimes n}$ is an E_R-comodule.

Let $M \subset N$ be finite dimension H_R-comodules. There exists $n \in \mathbb{N}$ such that $M \otimes {\Lambda^r_R}^{\otimes n}$ and $N \otimes {\Lambda^r_R}^{\otimes n}$ are E_R-comodules. Hence, there exists E_R-comodule P, such that:

$$N \otimes {\Lambda^r_R}^{\otimes n} = M \otimes {\Lambda^r_R}^{\otimes n} \oplus P.$$

Therefore

$$N = M \oplus P \otimes {\Lambda^r_R}^{*\otimes n}.$$

Thus every finite dimension H_R-comodule is absolutely reducible. Since every H_R-comodule is injective limit of a system of finite dimension comodules, (ii) is proved. (iii) also follows from this discussion.∎

In Section 4 part (i) of the Theorem will be generated to the case of arbitrary Hecke symmetry. But we emphases that Proposition 3.2 holds also for arbitrary (non-closed) Hecke operator.

Since minimal central idempotents of $\mathcal{H}_n$ are indexed by partitions of n, to any partition λ of n there associates a comodule M_λ which is either zero or simple comodule of H_R.

3.2. **The Integral on H_R.** By definition, a (left) integral on a Hopf algebra H_R is a linear functional $\int : H \longrightarrow k$ satisfying the following condition:

$$\int(x) = \sum_{(x)} x_1 \int(x_2). \tag{16}$$

If there exists an integral which does not vanish on the unit of H_R, then H_R is cosemi-simple. In this case we can rescale $\int$, such that $\int(1) = 1$ and call it a normalized integral. It is known that if a normalized integral exists then uniquely [26]. Our aim is to find such an integral for H_R.

Let us make H_R into a $\mathbb{Z}$-graded algebra setting $\deg_{\mathbb{Z}}(z^j_i) = 1, \deg_{\mathbb{Z}}(t^j_i) = -1$. For x homogeneous, $\Delta(x) = x_1 \otimes x_2$, then $\deg_{\mathbb{Z}}(x) = \deg_{\mathbb{Z}}(x_1) = \deg_{\mathbb{Z}}(x_2)$. Hence, (16) is not a homogeneous relation unless $\deg_{\mathbb{Z}}(x) = 0$, while H_R is homogeneous. Thus, we proved the following.

Lemma 3.4. *The value of $\int$ on a homogeneous element of H_R (with respect to the above grading) is equal to zero unless the degree of this element is zero.*

The problem reduces to finding the values of $\int$ on monomials in z^i_j's and t^k_l's, in which the numbers of z^i_j's and t^k_l's are equal. By virtue of the relation in (8), every monomials in z^i_j's and t^k_l's can be expressed as a linear combination of monomials in which z^i_j's are on the right of t^k_l's. Hence it reduces to finding:

$$\mathbf{I}^{n j_1, j_2, \ldots, j_n, l_1, l_2, \ldots, l_n}_{i_1, i_2, \ldots, i_n, k_1, k_2, \ldots, k_n} := \int (z^{j_1}_{i_1} z^{j_2}_{i_2} \cdots z^{j_n}_{i_n} t^{l_1}_{k_1} t^{l_2}_{k_2} \cdots t^{l_n}_{k_n}).$$

At the first sight, it is not clear if any set of values of $\int$ on $z^{j_1}_{i_1} z^{j_2}_{i_2} \cdots z^{j_n}_{i_n} t^{l_1}_{k_1} t^{l_2}_{k_2} \cdots t^{l_n}_{k_n}$ induces an integral on H_R, since there may be more then one way of expressing a monomial in z's and t's as linear combination of $z^{j_1}_{i_1} z^{j_2}_{i_2} \cdots z^{j_n}_{i_n} t^{l_1}_{k_1} t^{l_2}_{k_2} \cdots t^{l_n}_{k_n}$. Here the Yang-Baxter plays an important role, which ensures that different way of expressing yield the same result. In fact, it can be shown that there exists a bialgebra structure on $E_R \otimes E^{\mathrm{op,cop}}_R$ such that H_R is isomorphic as a Hopf algebra to a factor of $E_R \otimes E^{\mathrm{op,cop}}_R$ by some biideal (cf. [9], see also [12]). In this construction E_R is generated by z^j_i and $E^{\mathrm{op,cop}}_R$ by t^j_i, the biideal is generated by (7) and (11).

According to (16), we have:

$$z^{j_1}_{p_1} z^{j_2}_{p_2} \cdots z^{j_n}_{p_n} t^{m_1}_{k_1} t^{m_2}_{k_2} \cdots t^{m_n}_{k_n} \overset{n}{\mathbf{I}}{}^{p_1,p_2,\dots,p_n,l_1,l_2,\dots,l_n}_{i_1,i_2,\dots,i_n,m_1,m_2,\dots,m_n} = \overset{n}{\mathbf{I}}{}^{j_1,j_2,\dots,j_n,l_1,l_2,\dots,l_n}_{i_1,i_2,\dots,i_n,k_1,k_2,\dots,k_n}.$$

By virtue of (8), this is equivalent to

$$\overset{n}{\mathbf{I}}{}^{p_1,p_2,\dots,p_n,l_1,l_2,\dots,l_n}_{i_1,i_2,\dots,i_n,m_1,m_2,\dots,m_n} z^{j_1}_{p_1} z^{j_2}_{p_2} \cdots z^{j_n}_{p_n} = \overset{n}{\mathbf{I}}{}^{j_1,j_2,\dots,j_n,l_1,l_2,\dots,l_n}_{i_1,i_2,\dots,i_n,k_1,k_2,\dots,k_n} z^{k_n}_{m_n} \cdots z^{k_2}_{m_2} z^{k_1}_{m_1}. \tag{17}$$

For fixed $i_1, i_2, \dots, i_n, l_1, l_2, \dots, l_n$, (17) is a relation on z's being considered as elements of H_R. By Theorem 4.1 E_R is injectively embedded in H_R, hence (17) is a relation for z's as elements of E_R. By virtue of (14), (15), $\overset{n}{\mathbf{I}}$, for fixed $i_1, i_2, \dots, i_n, l_1, l_2, \dots, l_n$, should represent an element of $\overline{\mathcal{H}_n}$. Thus, $\overset{n}{\mathbf{I}}$ can be written in the following form:

$$\overset{n}{\mathbf{I}}{}^{j_1,j_2,\dots,j_n,l_1,l_2,\dots,l_n}_{i_1,i_2,\dots,i_n,k_1,k_2,\dots,k_n} = \sum_{w \in \mathfrak{S}_n} \overset{n}{\Lambda}_w{}^{l_n,\dots,l_2,l_1}_{i_1,i_2,\dots,i_n} R_w{}^{j_1,j_2,\dots,j_n}_{k_n,\dots,k_2,k_1}, \tag{18}$$

where $R_w = \rho_n(T_w)$ as in (13), $\{\overset{n}{\Lambda}_w \mid w \in \mathfrak{S}_n\}$ is a set of matrices indexed by $w \in \mathfrak{S}_n$.

According to relation (5) on z^i_j, we have the following relation on $\overset{n}{\Lambda}_w$ (we omit indices for simplicity):

$$\sum_{w \in \mathfrak{S}_n} \overset{n}{\Lambda}_w R_i \otimes R_w = \sum_{w \in \mathfrak{S}_n} \overset{n}{\Lambda}_w \otimes R_i R_w, 1 \le i \le n-1. \tag{19}$$

Let denote $\varepsilon_i(w) := \frac{1}{2}(l(w) - l(v_i w) + 1)$, where $v_i = (i, i+1)$ is a basic transposition in $\mathfrak{S}_n$, $w \in \mathfrak{S}_n$. We have

$$R_i R_w = q^{\varepsilon_i(w)} R_{v_i w} - \varepsilon_i(w)(q-1) R_w.$$

Thus, (19) can be rewritten as:

$$\sum_{w \in \mathfrak{S}_n} \left[\overset{n}{\Lambda}_w (R_i - \varepsilon_i(w)(q-1)) - q^{1-\varepsilon_i(w)} \overset{n}{\Lambda}_{v_i w} \right] \otimes R_w = 0.$$

This equation would imply that

$$\overset{n}{\Lambda}_w (R_i - \varepsilon_i(w)(q-1)) - q^{1-\varepsilon_i(w)} \overset{n}{\Lambda}_{v_i w} = 0, \tag{20}$$

if the $R_w, w \in \mathfrak{S}_n$ were linear independent being elements of $\overline{\mathcal{H}_n}$, in other words, if ρ_n in (13) were injective. This is, unfortunately, not always the case. Nevertheless, we assume that $\overset{n}{\Lambda}_w$ satisfy (20) which is equivalent to

$$\overset{n}{\Lambda}_w = q^{-l(w)} \overset{n}{\Lambda} R_{w^{-1}}, \quad \overset{n}{\Lambda} := \overset{n}{\Lambda}_{\mathbf{1}}.$$

Whence (18) has the form:

$$\overset{n}{\mathbf{I}}{}^{j_1,j_2,\dots,j_n,l_1,l_2,\dots,l_n}_{i_1,i_2,\dots,i_n,k_1,k_2,\dots,k_n} = \sum_{w \in \mathfrak{S}_n} q^{-l(w)} (\overset{n}{\Lambda} R_{w^{-1}})^{l_n,\dots,l_2,l_1}_{i_1,i_2,\dots,i_n} R_w{}^{j_1,j_2,\dots,j_n}_{k_n,\dots,k_2,k_1}. \tag{21}$$

On the other hand, the relations (7) and (11) raise up the following relations on $\mathbf{I}$:

$$\delta^{i_n}_{l_1} \overset{n}{\mathbf{I}}{}^{j_1,j_2,\dots,j_n,l_1,l_2,\dots,l_n}_{i_1,i_2,\dots,i_n,k_1,k_2,\dots,k_n} = \delta^{j_n}_{k_1} \overset{n-1}{\mathbf{I}}{}^{j_1,j_2,\dots,j_{n-1},l_2,l_3,\dots,l_n}_{i_1,i_2,\dots,i_{n-1},k_2,k_3,\dots,k_n}, \tag{22}$$

$$C^{k_1}_{j_n} \overset{n}{\mathbf{I}}{}^{j_1,j_2,\dots,j_n,l_1,l_2,\dots,l_n}_{i_1,i_2,\dots,i_n,k_1,k_2,\dots,k_n} = C^{l_1}_{i_n} \overset{n-1}{\mathbf{I}}{}^{j_1,j_2,\dots,j_{n-1},l_2,l_3,\dots,l_n}_{i_1,i_2,\dots,i_{n-1},k_2,k_3,\dots,k_n}. \tag{23}$$

Substituting (23) into (21), we get

$$C^{l_1}_{i_n}{}^{n-1}\mathbf{I}^{j_1,j_2,\dots,j_{n-1},l_2,l_3,\dots,l_n}_{i_1,i_2,\dots,i_{n-1},k_2,k_3,\dots,k_n} = \sum_{w\in\mathfrak{S}_n} (\overset{n}{\Lambda} R_{w^{-1}})^{l_n,\dots,l_2,l_1}_{i_1,i_2,\dots,i_n} C^{k_1}_{j_n} R_w{}^{j_1,j_2,\dots,j_n}_{k_n,\dots,k_2,k_1}. \tag{24}$$

For $w\in\mathfrak{S}_n$ there exists unique $w_1\in\mathfrak{S}_{(n-1,1)}$ (the Young subgroup corresponding to the partition $(n-1,1)$ of n) and $n\ge k\ge 1$, such that $w=w_{n-k}\cdots v_{n-2}v_{n-1}w_1$. Considering w_1 as an element of $\mathfrak{S}_{n-1}$, we have, for $k\le n-1$,

$$\begin{aligned} C^{k_1}_{j_n} R_w{}^{j_1,j_2,\dots,j_n}_{k_n,\dots,k_2,k_1} &= (R_{n-k}\cdots R_{n-2})^{p_n,\dots,p_2}_{k_n,\dots,k_2} R_{n-1}{}^{qj_n}_{p_2k_1} C^{k_1}_{j_n} R_{w_1}{}^{j_1,j_2,\dots,j_n}_{p_n,\dots,p_3,q} \\ &= (R_{n-k}\cdots R_{n-1})^{p_n,\dots,p_2}_{k_n,\dots,k_2} R_{w_1}{}^{j_1,j_2\dots,j_{n-1}}_{p_n,\dots,p_3,p_2}, \end{aligned}$$

and for $k=n-1$,

$$C^{k_1}_{j_n} R_w{}^{j_1,j_2,\dots,j_n}_{k_n,\dots,k_2,k_1} = \mathrm{Tr}(C) R_{w_1}{}^{j_2,\dots,j_n}_{k_n,\dots,k_2}.$$

For an operator $R_w=R_{n-k}\cdots R_{n-2}R_{n-1}R_{w_1}, 0\le k\le n-1, w_1\in\mathfrak{S}_{(n-1,1)}$ ($k=0$ means $w\in\mathfrak{S}_{(n-1,1)}$), set

$$\mathrm{Tr}^n_{q,c}(R_w) := \left[\begin{array}{l} R_{n-k}\cdots R_{n-2}R_{w_1} \quad \text{if } k\ge 1; \\ cR_{w_1} \text{ if } k=0 \end{array}\right. \tag{25}$$

and extend it linearly to a map $\mathrm{Tr}_{q,c}(R_w)^n:\mathcal{H}_n\longrightarrow\mathcal{H}_{n-1}$. Then we have, for $c=\mathrm{Tr}(C)$,

$$C^{k_1}_{j_n} R_w{}^{j_1,j_2,\dots,j_n}_{k_n,\dots,k_2,k_1} = \mathrm{Tr}_{q,c}(R_w)^n{}^{j_1,j_2,\dots,j_n}_{k_n,\dots,k_2,k_1}. \tag{26}$$

The following property of the operation $\mathrm{Tr}^n_{c,q}$ follows immediately from its definition and (26):

$$\rho_{n-1}(\mathrm{Tr}^n_{q,c}(h)) = 0 \text{ if } \rho_n(h)=0. \tag{27}$$

We emphases that (27) would not be true if R were not closed. We see that two even Hecke symmetries induce the same operator Tr_q iff they are of the same rank.

Let $c=\mathrm{Tr}(C)$, then the right hand side of (24) is equal to (indices are omitted)

$$\begin{aligned} \sum_{w\in\mathfrak{S}_n} q^{-l(w)} \overset{n}{\Lambda} R_{w^{-1}}\otimes \mathrm{Tr}^n_{q,c}(R_w) &= \sum_{w\in\mathfrak{S}_{(n-1,1)}} \mathrm{Tr}(C) q^{-l(w)} \overset{n}{\Lambda} R_{w^{-1}}\otimes R_w \\ &+ \sum_{\substack{w\in\mathfrak{S}_{(n-1,1)} \\ k\ge 1}} q^{-l(w)-k} \overset{n}{\Lambda} R_{w^{-1}}R_{n-1}\cdots R_{n-k}\otimes R_{n-k}\cdots R_{n-2}R_w. \end{aligned}$$

In the left hand side of the equation one should have in mind the identity $\mathfrak{S}_{n-1}\equiv\mathfrak{S}_{(n-1,1)}$, thus in the sum, w^{-1} runs in $\mathfrak{S}_{(n-1,1)}$ while w runs in $\mathfrak{S}_{n-1}$. Let us first examine the second term of the right-hand side of this equation for k fixed (omitting $\overset{n}{\Lambda}$ for the moment):

$$\sum_{w\in\mathfrak{S}_{(n-1,1)}} q^{-l(w)-k}R_{n-k}\cdots R_{n-1}R_{w^{-1}}\otimes R_wR_{n-2}\cdots R_{n-k}$$

$$= \sum_{\substack{w\in\mathfrak{S}_{(n-1,1)}\\ l(wv_{n-2})=l(w)+1}} \left[\begin{array}{r} q^{-l(w)-k}R_{n-k}\cdots R_{n-2}R_{n-1}R_{w^{-1}}\otimes R_{wv_{n-2}}R_{n-3}\cdots R_{n-k}\\ +q^{-l(w)-k-1}R_{n-k}\cdots R_{n-2}R_{n-1}R_{n-2}R_{w^{-1}}\otimes R_w(R_{n-2})^2R_{n-3}\cdots R_{n-k}\end{array}\right]$$

$$= \sum_{\substack{w\in\mathfrak{S}_{(n-1,1)}\\ l(wv_{n-2})=l(w)+1}} q^{-l(w)-k-1}\left[\begin{array}{r} qR_{n-k}\cdots R_{n-2}R_{n-1}R_{w^{-1}}\otimes R_{wv_{n-2}}R_{n-3}\cdots R_{n-k}\\ +qR_{n-1}R_{n-k}\cdots R_{n-2}R_{n-1}R_{w^{-1}}\otimes R_wR_{n-3}\cdots R_{n-2}\\ +(q-1)R_{n-k}\cdots R_{n-2}R_{w^{-1}}\otimes R_{wv_{n-2}}R_{n-3}\cdots R_{n-k}\end{array}\right]$$

$$= \sum_{\substack{w\in\mathfrak{S}_{(n-1,1)}\\ l(wv_{n-2})=l(w)+1}} q^{-l(w)-k-1}\left[\begin{array}{r} R_{n-k}\cdots R_{n-1}(R_{n-2})^2R_{w^{-1}}\otimes R_{wv_{n-2}}R_{n-3}\cdots R_{n-2}\\ +qR_{n-1}R_{n-k}\cdots R_{n-2}R_{n-1}R_{w^{-1}}\otimes R_wR_{n-3}\cdots R_{n-k}\end{array}\right]$$

$$= \sum_{\substack{w\in\mathfrak{S}_{(n-1,1)}\\ l(wv_{n-2})=l(w)+1}} \left[\begin{array}{r} q^{-l(wv_{n-2})-k}R_{n-1}R_{n-k}\cdots R_{n-2}R_{n-1}R^{-1}_{wv_{n-2}}\otimes R_{wv_{n-2}}R_{n-3}\cdots R_{n-k}\\ +q^{-l(w)-k}R_{n-1}R_{n-k}\cdots R_{n-1}R_{w^{-1}}\otimes R_wR_{n-3}\cdots R_{n-k}\end{array}\right]$$

$$= \sum_{w\in\mathfrak{S}_{(n-1,1)}} q^{-l(w)-k}R_{n-1}R_{n-k}\cdots R_{n-2}R_{n-1}R_{w^{-1}}\otimes R_wR_{n-3}\cdots R_{n-k}$$

$$= \cdots \text{ (repeating the above process } k-2 \text{ times)}$$

$$= \sum_{w\in\mathfrak{S}_{(n-1,1)}} q^{-l(w)-k}R_{n-1}R_{n-2}\cdots R_{n-k}R_{n-k+1}\ldots R_{n-2}R_{n-1}R_{w^{-1}}\otimes R_w$$

Thus, denoting

$$\overline{L_n} := q^{-1}R_{n-1}+q^{-2}R_{n-2}R_{n-1}R_{n-2}+\cdots+q^{-n+1}R_{n-1}R_{n-2}\cdots R_1R_2\cdots R_{n-1},\qquad n\geq 2,$$

$\overline{L_1}:=0$, and $c:=\mathrm{Tr}(C)$, we have

$$\sum_{w\in\mathfrak{S}_n} q^{-l(w)}R_{w^{-1}}\otimes \mathrm{Tr}^n_{q,c}(R_w) = \sum_{w\in\mathfrak{S}_{(n-1,1)}} q^{-l(w)}(c+\overline{L_n})R_{w^{-1}}\otimes R_w. \tag{28}$$

Remark. The above manipulation also applies for $T_w\in\mathcal{H}_n$ which yields (28) for R_w replaced by T_w.

Also (24) has the following form:

$$\sum_{w\in\mathfrak{S}_{(n-1,1)}} q^{-l(w)}(\overset{n-1}{\Lambda}\otimes C)R_{w^{-1}}\otimes R_w = \sum_{w\in\mathfrak{S}_{(n-1,1)}} q^{-l(w)}\overset{n}{\Lambda}(c+\overline{L_n})R_{w^{-1}}\otimes R_w.$$

Thus, if $R_w, w\in\mathfrak{S}_{n-1}$ are linear independent, then we can conclude that

$$\overset{n-1}{\Lambda}\otimes C = \overset{n}{\Lambda}(c+\overline{L_n}). \tag{29}$$

Since we assume that $\int(1) = 1$, we have $\overset{0}{\Lambda} = 1$. Hence, provided that $\overline{L_k} + c, 1 \leq k \leq n$ are invertible, we have

$$\overset{n}{\Lambda} = C^{\otimes n} \prod_{k=1}^{n} (\overline{L_k} + c)^{-1}. \tag{30}$$

Remark, that $\overline{L_k}$ is the images of the Murphy operators L_k through ρ (see Appendix), they commute since the Murphy operators commute.

Let R be be an even Hecke symmetry of rank r, then $c = -[-r]_q$ [7]. Hence $(c + \overline{L_n})$ is not invertible in $\mathcal{H}_n$ for $n > r$. Fortunately, Lemma A.3 shows that $(c + \overline{L_n})$ is invertible in $\overline{\mathcal{H}_n}$, thus (30) always has meaning. It remains to check the relation (22) for $\overset{n}{\mathbf{I}}$. By virtue of (21) and (29), (22) has the form:

$$\sum_{w \in \mathfrak{S}_n} q^{-l(w)} \delta^{i_n}_{l_1} \left[(\overset{n-1}{\Lambda} \otimes C)(c + \overline{L_n})^{-1} R_{w^{-1}} \right]^{l_n, \dots, l_2, l_1}_{i_1, i_2, \dots, i_n} R_w{}^{j_1, j_2, \dots, j_n}_{k_n, \dots, k_2, k_1}$$
$$= \sum_{w \in \mathfrak{S}_{n-1}} q^{-l(w)} \delta^{j_n}_{k_1} (\overset{n-1}{\Lambda} R_{w^{-1}})^{l_n, \dots, l_2, l_1}_{i_1, i_2, \dots, i_n} R_w{}^{j_1, \dots, j_{n-1}}_{k_n, \dots, k_2}. \tag{31}$$

According to (26), we have:

$$\delta^{i_n}_{l_1} \left[(\overset{n-1}{\Lambda} \otimes C)(c + \overline{L_n})^{-1} R_{w^{-1}} \right]^{l_n, \dots, l_2, l_1}_{i_1, i_2, \dots, i_n} = \left[\overset{n-1}{\Lambda} \operatorname{Tr}_{q,c}((c + \overline{L_n})^{-1} R_{w^{-1}}) \right]^{l_n \dots, l_2}_{i_1, \dots i_{n-1}}.$$

Thus, (31) can be rewritten as:

$$\sum_{w \in \mathfrak{S}_n} q^{-l(w)} \overset{n-1}{\Lambda} \operatorname{Tr}^n_{q,c} \left((c + \overline{L_n})^{-1} R_{w^{-1}} \right) \otimes R_w = \sum_{w \in \mathfrak{S}_{(n-1,1)}} q^{-l(w)} \overset{n-1}{\Lambda} R_{w^{-1}} \otimes R_w.$$

Using the method of the proof of Equation (28) we can show the following equation:

$$\sum_{w \in \mathfrak{S}_n} q^{-l(w)} \operatorname{Tr}^n_{q,c}(T_{w^{-1}}) \otimes T_w = \sum_{w \in \mathfrak{S}_{(n-1,1)}} q^{-l(w)} T_{w^{-1}} \otimes T_w (c + L_n). \tag{32}$$

Let $p \in \mathcal{H}_n$ such that $\overline{p} = \rho(p)$ satisfies $\overline{p}(c + \overline{L_n}) = \mathbf{1}_{\overline{\mathcal{H}}_n}$. Let

$$(c + \overline{L_n}) T_v = \sum_{u \in \mathfrak{S}_n} c^u_v T_u \text{ and } p T_v = \sum_{u \in \mathfrak{S}_n} d^u_v T_u,$$

then

$$\sum_{w \in \mathfrak{S}_n} c^w_v d^u_w R_u = \delta_{u,v} R_v.$$

Set $< T_u, T_v >= \delta_{u,v} q^{l(v)}$ and extend it linearly on $\mathcal{H}_n$, thus we get a non-degenerate scalar product on $\mathcal{H}_n$ which satisfies $< hk, g >=< h, gk^* >$, where $*$ is the linear extension of the map $T_w \longmapsto T_{w^{-1}}, w \in \mathfrak{S}_n$ ([1], Lemma 2.2). Since $(c + L_n)$ is $*$-invariant, we have:

$$q^{l(w)} c^w_v =< T_w, (c + L_n) T_v >=< (c + L_n) T_w, T_v > .$$

Hence

$$(c + L_n) T_w = \sum_{v \in \mathfrak{S}_n} q^{l(w) - l(v)} c^w_v T_v. \tag{33}$$

Since T_w, $w \in \mathfrak{S}_n$ form a basis for $\mathcal{H}_n$, we can conclude that $c_v^u = q^{l(v)-l(u)} c_u^v$. Applying operation $*$ on (33) we get

$$R_{w^{-1}}(c+L_n) = \sum_{v\in\mathfrak{S}_n} q^{l(w)-l(v)} c_v^w R_{v^{-1}}.$$

Now, we have

$$\begin{aligned}
&\sum_{w\in\mathfrak{S}_n} q^{-l(w)} \mathrm{Tr}^n_{q,c}((c+\overline{L_n})^{-1} R_{w^{-1}}) \otimes R_w(c+\overline{L_n}) \\
&= \sum_{w,u,v\in\mathfrak{S}_n} q^{-l(w)} d^{v^{-1}}_{w^{-1}} \mathrm{Tr}^n_{q,c}(R_{v^{-1}}) \otimes q^{l(w)-l(u)} c^{w^{-1}}_{u^{-1}} R_u \\
&= \sum_{w,u,v\in\mathfrak{S}_n} q^{-l(u)} d^{v^{-1}}_{w^{-1}} \mathrm{Tr}^n_{q,c}(R_{v^{-1}}) c^{w^{-1}}_{u^{-1}} \otimes R_u \\
&= \sum_{u\in\mathfrak{S}_n} q^{-l(u)} \mathrm{Tr}^n_{q,c}(R_{u^{-1}}) \otimes R_u \\
\text{(by virtue of (32))} &= \sum_{u\in\mathfrak{S}_n} q^{-l(v)} (R_{v^{-1}}) \otimes R_v(c+\overline{L_n}).
\end{aligned}$$

Since $(c+\overline{L_n})$ is invertible, (22) is proved. Thus, we have

Theorem 3.5. *Let R be an even Hecke symmetry of rank r. Then the integral of H_R can be given by the following formula*

$$\overset{n}{\mathbf{I}} = \sum_{w\in\mathfrak{S}_n} q^{-l(w)} \left(\prod_{k=1}^{n} (\overline{L_k}+c)^{-1}\right) C^{\otimes n} R_{w^{-1}} \otimes R_w. \tag{34}$$

3.3. **Structure of H_R-Comod.** We now study the fusion rules for H_R-Comod. Let $\lambda \vdash n$ and $\mu \vdash m$ and M_λ and M_μ be the associated simple comodules of E_R, and hence simple comodules of H_R. The tensor product of M_λ and M_μ decomposes into direct sum of simple ones:

$$M_\lambda \otimes M_\mu \cong \bigoplus_\gamma c^\gamma_{\lambda,\mu} M_\gamma,$$

where $c^\gamma_{\lambda,\mu}$ denotes the multiplicity of M_γ entering the above decomposition. $c^\gamma_{\lambda,\mu}$ are called fusion of Clebsch-Gordan coefficients. Our aim is to show that these coefficients depend only on the rank of R. Let e_λ and e_μ be primitive idempotents of $\mathcal{H}_n$ and $\mathcal{H}_m$ respectively, $M_\lambda = \rho_n(e_\lambda)$, $M_\mu = \rho_m(e_\mu)$. Embed $\mathcal{H}_n$ into $\mathcal{H}_{n+m}$, $T_i \longmapsto T_i$ and $\mathcal{H}_n$ into $\mathcal{H}_{n+m}$, $T_j \longmapsto T_{j+n}$. Then $M_\lambda \otimes M_\mu \cong \mathrm{Im}(\rho_{n+m}(e_\lambda e_\mu^n))$, where e_μ^n is the image of e_μ under the defined above embedding. The decomposition of $e_\lambda e_\mu^n$ into primitive idempotents depends only on λ and μ themselves. In particular, let $\lambda = (\lambda_1\lambda_2,\ldots,\lambda_k)$, then we have the decomposition

$$M_{(\lambda_1)} \otimes M_{(\lambda_2)} \otimes \cdots \otimes M_{(\lambda_k)} \cong \sum_{\mu\vdash n} m_{\lambda\mu} M_\mu, \tag{35}$$

with the coefficients $m_{\lambda\mu}M_\mu$ as in the case of representations of $\mathrm{GL}(n)$. They are given by the formula (cf. [13], Section 6)

$$[\lambda_1][\lambda_2]\cdots[\lambda_k] = \sum m_{\lambda\mu}[\mu].$$

Hence we have the determinant form for M_λ (cf. [13], Section 19):

$$M_\lambda \cong \det\left|\left(M_{(\lambda_i-i+j)}\right)_{1\le i,j\le k}\right|. \tag{36}$$

Hence the problem reduces to investigating when $\rho_n(e_\lambda) = 0$.

It is known that E_R and H_R are coquasitriangular bialgebras (cf. [17, 25]). Let τ denote the braiding on H_R-Comod induced from the coquasitriangular structure on H_R. In particular, $\tau_{V,V} = R, \tau_{V^*,V} = R_{V^*,V}, \tau_{V,V^*} = R_{V,V^*}$. Let M be an object in H_R-Comod, we define its 8-dim to be [21]:

$$8\text{-dim}(M) := \mathrm{ev}_M \tau_{M,M^*} \mathrm{db}_M(\mathbf{1}).$$

For example $8\text{-dim}(V) = \mathrm{Tr}(C)$. More general, for a morphism $f : M \longrightarrow M$ we define its 8-Tr to be

$$8\text{-Tr}(f) := \mathrm{ev}_M \tau_{M,M^*}(f \otimes M^*)\mathrm{db}_M(\mathbf{1}_k).$$

Then we have

$$8\text{-dim}(M) = 8\text{-Tr}(\mathrm{id}_M), \tag{37}$$
$$8\text{-Tr}(fg) = 8\text{-Tr}(gf). \tag{38}$$

Lemma 3.6. *Let M be a simple H_R-comodule, then* $8\text{-dim}(M) \neq 0$.

Proof. Since M is simple,

$$\mathrm{Hom}(M^* \otimes M, k) \cong \mathrm{Hom}(M, M) = k. \tag{39}$$

$\mathrm{db}_M : k \longrightarrow M \otimes M^*$ is not trivial, hence induces a morphism $p : M \otimes M^* \longrightarrow k$, such that $p \circ \mathrm{db}_M = \mathbf{1}_k$. Thus $p \neq 0$, hence $0 \neq p \circ \tau^{-1}_{M,M^*} : M^* \otimes M \longrightarrow k$. By (39) there exists $c \in k^\times$, such that $p \circ \tau^{-1}_{M,M^*} = \mathrm{ev}_M$, hence $\mathrm{ev}_M \tau \mathrm{db} = c \neq 0$.∎

Let $M = M_1 \oplus M_2$ with morphisms $p_i : M \longrightarrow M_i$, $j_i : M_i \longrightarrow M$, $p_i \circ j_i = \mathbf{1}_{M_i}$, $j_1p_1 + j_2p_2 = \mathbf{1}_M$. Then

$$\begin{aligned} 8\text{-dim}(M) = &\ \mathrm{db}_M \tau_{M,M^*}(1 \otimes (j_1p_1 + j_2p_2))\mathrm{ev}_M \\ = &\ \textstyle\sum_i \mathrm{db}_M \tau_{M,M^*}(1 \otimes j_ip_i)\mathrm{ev}_M \\ = &\ \textstyle\sum_i \mathrm{db}_M \tau_{M,M^*}(j_i^* \otimes p_i)\mathrm{ev}_{M_i} \\ = &\ \textstyle\sum_i \mathrm{db}_M (p_i \otimes j_i^*)\tau_{M_i,M_i^*}\mathrm{ev}_{M_i} \\ = &\ \mathrm{db}_{M_i}((j_1p_1 + j_2p_2) \otimes 1)\tau_{M_i,M_i^*}\mathrm{ev}_{M_i} \\ = &\ 8\text{-dim}(M_1) + 8\text{-dim}(M_2). \end{aligned}$$

Thus the 8-dim is additive.

Let us fix the following standard rule of defining the dual to the tensor product. Let V_i, $i = 1, 2, \ldots, k$ be H_R-comodules, then

$$(V_1 \otimes V_2 \otimes \cdots \otimes V_k)^* := V_k^* \otimes V_{k-1}^* \otimes \cdots \otimes V_1^*,$$

with the evaluation map $\mathrm{ev} = \mathrm{ev}_k \mathrm{ev}_{k-1} \cdots \mathrm{ev}_1$. Let $M = \mathrm{Im}(\rho(e))$ be a simple H_R-comodules, e is a primitive idempotent in H_R. Considering e as a morphism of H_R-comodules, since $e = e^2$, we have

$$\begin{aligned} 8\text{-dim}(M) = &\ \mathrm{db}_{V^{\otimes n}}(e \otimes e^*)\tau_{V^{\otimes n},V^{\otimes n*}}\mathrm{ev}_{V^{\otimes n}}(\mathbf{1}_k) \\ = &\ \mathrm{db}_{V^{\otimes n}}(e \otimes V^{\otimes n^*})\tau_{V^{\otimes n},V^{\otimes n*}}\mathrm{ev}_{V^{\otimes n}}(\mathbf{1}_k) \\ = &\ 8\text{-Tr}(e) \\ = &\ \mathrm{Tr}^1_{q,c}(\mathrm{Tr}^2_{q,c}(\cdots(\mathrm{Tr}^n_{q,c}(T_{w_n}{}^{-2}e))\cdots)) \end{aligned} \tag{40}$$

where w_n is the longest element of $\mathfrak{S}_n$, that reverses the order of the elements $1, 2, \ldots, n$. The last equation above can be best explained using braid representation of τ (see the Appendix). Thus we can consider 8-dim(M) as an element of H_0 acting on $V^{\otimes 0} = k$.

By induction we can express w_n as $w_n = w_{n-1}v_{n-1}\dots v_2v_1$, hence can easily check that $(T_{w_n})^2 = q^{n-1}(1+(q-1)L_n)T_{w_{n-1}}$. Therefore

$$(T_{w_n})^2 = \prod_{k=1}^{n} q^{k-1}(1+(q-1)L_k). \tag{41}$$

Hence, assuming that $e = e_{i,\lambda}$, $\lambda \vdash n$ as in Lemma A.2, (ii), and using (65), we have

$$8\text{-dim}(M) = q^{-s(\lambda)-n(n-1)/2}\mathrm{Tr}^1_{q,c}(\mathrm{Tr}^2_{q,c}(\cdots(\mathrm{Tr}^n_{q,c}(e))\cdots)), \tag{42}$$

where $s(\lambda) := \sum_{k=1}^{n} r_\lambda(k)$ (see Appendix, Theorem A.1), which depends only on λ but not the λ-tableaux.

Proposition 3.7. *Let R be an even Hecke symmetry of rank r. Let $\lambda \vdash n$ and $M = \mathrm{Im}\rho_n(e_\lambda)$. Then $M_\lambda \neq 0$ (and hence is a simple comodule of H_R) iff $\lambda_{r+1} = 0$.*

Proof. By virtue of Lemma 3.6, it is sufficient to show that

$$\mathrm{Tr}^{1,2,\dots,n}_{q,r}(e) := \mathrm{Tr}^1_{q,r}(\mathrm{Tr}^2_{q,r}(\cdots(\mathrm{Tr}^n_{q,r}(e))\cdots)) \neq 0$$

iff $\lambda_{r+1} = 0$. Let R^r be the Drinfel'd-Jimbo R-matrix of type A_{r-1}, defined in (3), then R^r has rank r. Let ρ^q_n denote the representation of $\mathcal{H}_n$ induced by R^r. If $\lambda_{r+1} = 0$ then $V^{\otimes n}$, being considered as H_n-module, contains direct summand isomorphic to the simple $\mathcal{H}_n$-module S_λ (see Appendix) ([3], Proposition 5.1). On this module $e_\lambda \neq 0$, hence $\mathrm{Im}(\rho_n(e_\lambda)) \neq 0$. By Lemma 3.6 and (42), $\mathrm{Tr}^{1,2,\dots,n}_{q,r}(e_\lambda) \neq 0$.

If $\lambda_{r+1} \neq 0$, then $V^{\otimes n}$, being considered as $\mathcal{H}_n$-module through ρ^q_n, does not contains direct summand that isomorphic to S_λ as $\mathcal{H}_n$-module, so that e_λ acts as zero on $V^{\otimes n}$. Hence $\mathrm{Tr}^{1,2,\dots,n}_{q,r}(e_\lambda)$ acts as zero on $V^{\otimes 0} = k$, by (25). On the other hand, $\mathrm{Tr}^{1,2,\dots,n}_{q,r}(e_\lambda)$ is a constant, hence it is equal to zero.∎

Theorem 3.8. *Let R be an even Hecke symmetry of rank r. Let $\lambda \vdash n$, $\mu \vdash m$, then we have*

$$M_\lambda \otimes M_\mu \cong \bigoplus_{\substack{\gamma\vdash(n+m)\\ \gamma_{r+1}=0}} c^\gamma_{\lambda,\mu} M_\gamma. \tag{43}$$

Hence, the category H_R-Comod is equivalent to the category of representation of the standard quantum group $GL_q(r)$.

3.4. **The** 8-dim **of simple comodules.** The 8-dim is an invariant in H_R-Comod, however it is not tensor multiplicative and hence is difficult to compute for non-simple comodules. Even the formula for 8-dim$(V^{\otimes n})$ is very complicated. We provide in this section the so called ribbon-dimesion – rdim – which is multiplicative and, on simple comodules, it differs from 8-dim only by a power of q. In fact, for a simple E_R-comodule M of degree n, defined by a primitive idempotent $e \in \mathcal{H}_n$, its ribbon-dimension is by definition

$$\mathrm{rdim}(M) := \mathrm{Tr}^{1,2,\dots,n}_{q,c}(e). \tag{44}$$

Since E_R-Comod is semi-simple, one can extend (44) linearly on all E_R-comodules. To show that rdim is well defined, one has to verify that the right hand side of (44) is independent of the choice of the idempotent e. Before showing this, let us make it clear the meaning of ribbon-dimension.

By definition, a ribbon in a strict braided category is a natural isomorphism θ subject to the following equations:

$$\begin{aligned}\theta_{M\otimes N} &= (\theta_M \otimes \theta_N)\tau_{N,M}\tau_{M,N},\\ \theta_{M^*} &= (\theta(M))^*,\end{aligned} \tag{45}$$

where τ denotes the braiding in the category. This definition first appeared in the works of Joyal and Street [15] and Turaev [29]. A strict braided rigid category with a ribbon is called ribbon category. For more details on ribbon categories, the reader is referred to a book of C. Kassel [16]. For an object M in a ribbon category, the ribbon-dimension, $\mathrm{rdim}(M)$, is by definition the morphism

$$\mathrm{ev}_M\tau_{M,M^*}(\theta_M\otimes M^*)\mathrm{db}_M : I \longrightarrow I. \tag{46}$$

By virtue of (45), rdim is multiplicative (cf. [16], Chapter XIV).

Lemma 3.9. (i) E_R-Comod *possesses a ribbon. Let e be an idempotent in $\mathcal{H}_n$, and $M = \mathrm{Im}(\rho(e))$, then the ribbon on M is given by*

$$\theta_M := {T_{w_n}}^2 e. \tag{47}$$

(i) *θ extends to a ribbon on H_R-*Comod*, hence H_R-*Comod *is a ribbon category.*

Proof. (i) follows from the fact that ${T_{w_n}}^2$ commutes with all elements of $\mathcal{H}_n$. This can be most easily seen from (41) in which ${T_{w_n}}^2$ is a symmetric polynomials on $L_1, L_2, \ldots, L_n$, hence belongs to the center of $\mathcal{H}_n$ according to Theorem A.1, (i). One can also verify directly that ${T_{w_n}}^2T_i = T_i{T_{w_n}}^2$.

(ii) follows from Theorem 3.3, (iii). One sets $\theta_{\Lambda^{r*}} = (\theta_{\Lambda^r})^*$ and uses (46).∎

Corollar 3.10. *(44) is well defined and* rdim *is tensor multiplicative.*

Proof. Let $e_1, e_2 \in \mathcal{H}_n$ define the same comodule, then they belong to the same minimal two-sided ideal of $\mathcal{H}_n$. Hence they are conjugate, $e_2 = ae_1a^{-1}$. Thus, we have

$$\begin{aligned}
\mathrm{Tr}_{q,c}^{1,2,\ldots,n}(e_2) &= \text{8-}\mathrm{Tr}({T_{w_n}}^2e_2)\\
&= \text{8-}\mathrm{Tr}({T_{w_n}}^2ae_1a^{-1})\\
&= \text{8-}\mathrm{Tr}(a^{-1}{T_{w_n}}^2ae_1)\\
&= \text{8-}\mathrm{Tr}({T_{w_n}}^2e_1)\\
&= \mathrm{Tr}_{q,c}^{1,2,\ldots,n}(e_1)
\end{aligned}$$

Hence, according to (36), we have the formula for the rdim of M_λ:

$$\mathrm{rdim}(M_\lambda) = \det\left(\mathrm{rdim}(M_{(\lambda_i-i+j)})\right)_{1\le i,j\le r}. \tag{48}$$

Direct computation shows that

$$\mathrm{rdim}(M_{(k)}) = \mathrm{Tr}_{q,c}^{1,2,\ldots,k}(e_{(k)}) = q^{-kr}\frac{[k+r-1]_q}{[r-1]_q![k]_q!}. \tag{49}$$

The proof of the following proposition will be given in the Appendix.

Proposition 3.11. *Let M_λ be a simple E_R-comodule corresponding to a partition $\lambda \vdash n$. Then*

$$\mathrm{rdim}(M_\lambda) = q^{-\sum_{i=1}^r\lambda_i(r+1-i)}\prod_{1\le i<j\le r}\frac{[\lambda_i-\lambda_j+j-i]_q}{[j-i]_q}. \tag{50}$$

Corollar 3.12. *With the assumption of Proposition 3.11*

$$\text{8-}\mathrm{dim}(M_\lambda) = q^{-\frac{1}{2}(n^2+\sum_{i=1}^r\lambda_i(\lambda_i+2r-4i+2))}\prod_{1\le i<j\le r}\frac{[\lambda_i-\lambda_j+j-i]_q}{[j-i]_q}. \tag{51}$$

In order to extend the formula (51) to a formula for all H_R-comodules we need the following lemma.

Lemma 3.13. *Let M, N be E_R-comodules, and N^* be the left dual comodule to N, then*

$$8\text{-dim}(M\otimes N^*) = \mathrm{ev}_{N\otimes M}\tau_{N\otimes M, M^*\otimes N^*}((\tau_{M,N}\tau_{N,M})^2\otimes M^*\otimes N^*)\mathrm{db}_{N\otimes M}. \tag{52}$$

Proof. The proof is based on the fact that N is isomorphic to the left dual of N^* by means of the brading. In fact, we can choose $\mathrm{ev}_{N^*} = \tau_{N,N^*}\mathrm{ev}_N$, $\mathrm{db}_{N^*} = \mathrm{db}_N{\tau_{N,N^*}}^{-1}$. ∎

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\lambda_i \in \mathbb{Z}$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$. Let $\lambda_0 := \mathrm{Min}\{\lambda_i\}$, $\overline{\lambda}_i := \lambda_i - \lambda_0$. Define

$$M_\lambda := M_{\overline{\lambda}} \otimes \Lambda^{r\otimes\lambda_0},$$

where $\overline{\lambda} = (\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_r)$, and

$$\Lambda^{r\otimes\lambda_0} := \left[\begin{array}{ll} \Lambda^{r\otimes\lambda_0} & \text{if } \lambda_0 \geq 0; \\ \Lambda^{r*\otimes|\lambda_0|} & \text{otherwise.}\end{array}\right.$$

Theorem 3.14. *Let λ and M_λ be as above, then*

$$8\text{-dim}(M_\lambda) = q^{-\frac{1}{2}(n^2+\sum_{i=1}^r \lambda_i(\lambda_i+2r-4i+2))}\prod_{1\leq i<j\leq r}\frac{[\lambda_i-\lambda_j+j-i]_q}{[j-i]_q}, \tag{53}$$

where $n = \sum \lambda_i$.

Proof. If $\lambda_0 \geq 0$ then (53) is just (51). Let $\lambda_0 < 0$. According to Lemma 3.13, it is sufficient to show that

$$\tau_{V^{\otimes n},\Lambda^r}\tau_{\Lambda^r,V^{\otimes n}} = q^{(r+1)n}.$$

Using the equality ${\tau_{U,V\otimes W}}^2 = (\tau_{U,V}\otimes W)(V\otimes {\tau_{U,W}}^2)(\tau_{U,V}\otimes W)$ one reduces to the case $n = 1$. In this case, ${\tau_{\Lambda^r,V}}^2 = R_rR_{r-1}\cdots R_1R_1R_2\cdots R_r = q^r(1+(q-1)\overline{L_r})$. We have $\Lambda^r = \mathrm{Im}(\rho(y_r))$ and $q^ry_r(1+(q-1)\overline{L_r}) = q^{r+1}$, that proves the equation above. Equation (53) can then be derived by easy computation. ∎

We see that 8-dim is an invariant of the category H_R-Comod up to a scalar constant. In fact if we make the transformation $R \longmapsto k\cdot R$, which also deforms the Hecke equation, then 8-dim will be multiplied by k^{-n^2}, where n is the homogeneous degree of the comodule. In particular we have

$$8\text{-dim}(M_{\underbrace{(1,1,\ldots,1)}_{r}}) = 8\text{-dim}(\Lambda^r) = q^{-r(r+1)/2}.$$

Further we remark that $M_\lambda\otimes\Lambda^r$ and M_λ, $\lambda\vdash n$, have different 8-dim:

$$8\text{-dim}(M_\lambda\otimes\Lambda^r) = q^{-(r+2n)(r+1)/2}8\text{-dim}(M_\lambda).$$

Assume now that the quantum determinant commutes with elements of H_R (see [7] for the necessary and sufficient condition). Then we can set it equal to 1 to get the special matrix quantum group SH_R. In this case, the comodule Λ^r, being considered as a SH_R-comodule is isomorphic to the trivial module. Hence, it is expected that this module has an 8-dim equal to 1. This can be made by rescaling $R \longmapsto q^{-\frac{r+1}{2r}}$. We call the rescaled 8-dim normalized 8-dim and denote it by $\overline{8\text{-dim}}$,

$$\begin{aligned}\overline{8\text{-dim}}(M_\lambda) &= q^{-\frac{1}{2}(\frac{n^2}{r}+\sum_{i=1}^r\lambda_i(\lambda_i+2r-4i+2))}\prod_{1\leq i<j\leq r}\frac{[\lambda_i-\lambda_j+j-i]_q}{[j-i]_q}\\ &= q^{-\frac{1}{2r}\sum_{1\leq i<,j\leq r}(\lambda_i-\lambda_j)^2-\sum_{i=1}^r\lambda_i(r-2i+1)}\prod_{1\leq i<j\leq r}\frac{[\lambda_i-\lambda_j+j-i]_q}{[j-i]_q}\end{aligned}$$

The last equation shows that the $\overline{\text{8-dim}}$ of M_λ and of $M_\lambda \otimes \Lambda^r$ are equal. As a corollary of the above discussion we have

Proposition 3.15. *Let R be such an even Hecke symmetry that the quantum determinant commutes with a elements of H_R, so that one can set it equal to 1 to get the corresponding special matrix quantum groups SH_R. Then the coquasitriangular structure $< \cdot|\cdot >$ on SH_R can be defined setting*

$$< z_i^k | z_j^l >= q^{-\frac{r+1}{2r}} R_{ij}^{lk}.$$

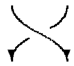
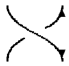
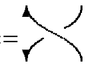
4. Non-Even Hecke Symmetry

We want to show in this section that the canonical homomorphism $i : E_R \longrightarrow H_R$ is injective if R is a Hecke symmetry, i.e., a closed Hecke operator. Then, using the method in Section 3.2 we give the criterion for non-semi-simplicity of H_R for non-even Hecke symmetry R.

Let $\mathcal{V}$ be the k-additive category, generated by V and R, that is, objects of $\mathcal{V}$ are $V^{\otimes n}$ and morphisms in $\mathcal{V}$ are obtained from R and the identity morphisms by composing, taking tensor products and linear combinations with coefficient in k. Analogously, let $\mathcal{W}$ be the k-additive monoidal category, generated by V, V^* and $R_{V,V}, R_{V^*,V}, R_{V,V^*}$ and ev_V, db_V. It this known that E_R (resp. H_R) is isomorphic to the Coend of the forgetful functor $v : \mathcal{V} \longrightarrow \mathrm{Vect}_k$ (resp. $w : \mathcal{W} \longrightarrow \mathrm{Vect}_k$) (cf. [25]). Relations on E_R (resp. H_R) are obtained from morphisms in $\mathcal{V}$ (resp. $\mathcal{W}$). In fact, the relations on E_R are obtained from $\mathrm{Hom}_{\mathcal{V}}(V^{\otimes n}, V^{\otimes m})$. To show that $i : E_R \longrightarrow H_R$ is injective, it is sufficient to show that

$$\mathrm{Hom}_{\mathcal{V}}(V^{\otimes n}, V^{\otimes m}) \cong \mathrm{Hom}_{\mathcal{W}}(V^{\otimes n}, V^{\otimes m}). \tag{54}$$

If $m \neq n$, then both side of (54) are empty, hence assume that $n = m$. Since R satisfies the Yang-Baxter equation, one can represent $R_{V,V}, R_{V,V^*}, R_{V^*,V}$ by oriented braids:

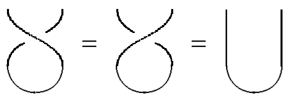
$R_{V,V} :=$  $R_{V,V^*} :=$  $R_{V^*,V} :=$ 

The morphisms ev_V and db_V are represented by oriented semi-circles:

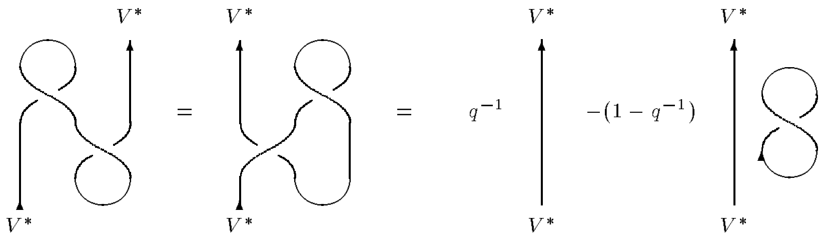
$\mathrm{ev}_V :=$  $\mathrm{db}_V :=$ 

Thus, a morphism in $\mathcal{W}$ is a linear combination of morphisms which can be represented by an oriented tangle (cf. [16]) in which a strand that connects V and V orientates itself upward and a strand that connect V^* and V^* orientates itself downward. Remark, that not every oriented tangle corresponds to a morphism, for example a circle does not, since the oriented semi-circles:

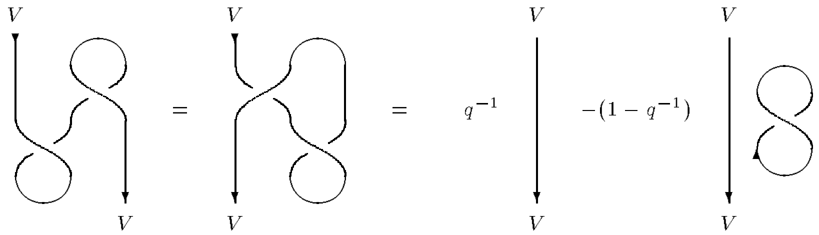
do not present any morphism. Furthermore, not every Reidermeister move is allowed, in fact, all except the second Reidermeister move:



are allowed. However, since R obeys the Hecke equation, we have

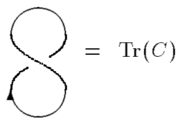


(55)

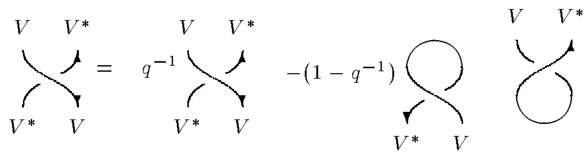


(56)

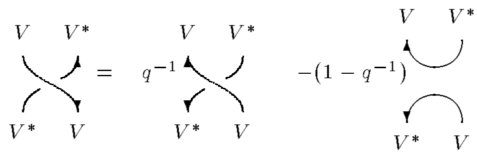
Using the notation of the previous section we have:



On the other hand we have:

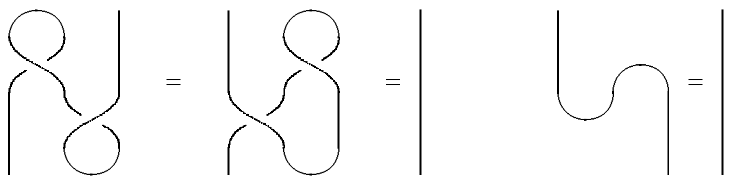


(57)



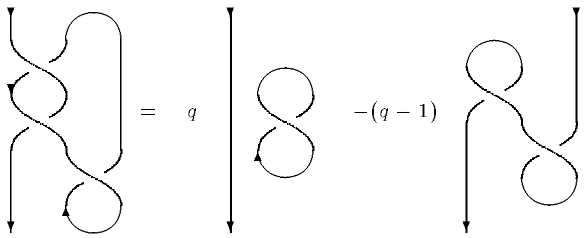
(58)

These equation follow immediately from the equations below:



(59)

A tangle which represents a morphism in $\mathrm{Hom}_{\mathcal{W}}(V^{\otimes n}, V^{\otimes n})$ may contain some strands which orientate themselves downward and some closed strands. Using (57) and (58) we can disjoint the closed strands, i.e., represent the tangle as a combination of tangle in which the closed strands a isolated, that is, the do not intersect with other strands. For example, we have:



An isolated closed strand represents a morphism of k, i.e., a scalar. Further, using (55), (56) and (59) we can transform a non closed strands into one, that does not have local extremals. This means, a tangle with isolated closed strands is equivalent to a braid up to a scalar. Thus, we see that a tangle which represents a morphism in $\mathrm{Hom}_{\mathcal{W}}(V^{\otimes n}, V^{\otimes n})$ is a linear combination of braids, hence the following equation holds

$$\mathrm{Hom}_{\mathcal{W}}(V^{\otimes n}, V^{\otimes n}) \cong \mathrm{Hom}_{\mathcal{V}}(V^{\otimes n}, V^{\otimes n}),$$

from which the theorem below follows immediately.

Theorem 4.1. *Let R be a closed Hecke operator then the canonical homomorphism $i : E_R \longrightarrow H_R$ is injective.*

If $\mathrm{Tr}(C)$ does not have the form $-[r]_q$ for some $r \in \mathbb{Z}$ then the proof of Theorem 3.5 shows that H_R is co-semi-simple. However this seems hardly to occur. In fact Equation (55) shows that $q^{-1} - (1 - q^{-1})\mathrm{Tr}(C) \neq 0$ for *all* q (also for roots of unity). Since the Yang-Baxter and the Hecke equations are algebraic equations on entries of R, their solutions form an algebraic variety. Hence, up to a change of basis in V we can assume that entries of R depend algebraic on q, i.e., they are rational functions on q, hence so is $\mathrm{Tr}(C)$. Therefore $\mathrm{Tr}(C)$ should have the form $-[r]_q$.

Theorem 4.2. *Let R be a closed Hecke symmetry and $Tr(B) = -[r]$ for some $r \in \mathbb{Z}$. If $\rho_{|r|+1}$ is injective, then H_R is not co-semi-simple.*

Proof. Let us assume that H_R is semi-simple, then the integral does exist. Since $\rho_{|r|+1}$ is injective, $\overset{|r|+1}{\mathbf{I}}$ should have the form (18) with $\overset{|r|+1}{\Lambda}$ satisfying (30). On the other hand, that $\rho_{|r|+1}$ is injective, implies that $\overline{L_{|r|+1}} + [r]$ in not invertible, a contradiction which proves the Theorem.∎

As the first application of this theorem we consider Takeuchi-Tambara's operator [28]:

$$\begin{aligned} R_\lambda = \; & [1 - \tfrac{(\lambda-1)^2}{2}]e_1^1 \otimes e_1^1 + [1 - \tfrac{(\lambda+1)^2}{2}]e_2^2 \otimes e_2^2 \\ & + \tfrac{(\lambda-1)^2}{2} e_1^2 \otimes e_1^2 + \tfrac{(\lambda+1)^2}{2} e_2^1 \otimes e_2^1 \\ & + \tfrac{1-\lambda^2}{2}(e_1^1 \otimes e_2^2 + e_2^2 \otimes e_1^1) \\ & + \tfrac{1+\lambda^2}{2}(e_1^2 \otimes e_1^2 + e_1^2 \otimes e_1^2). \end{aligned}$$

Let $q = 1/\lambda^2$ and $R_T = R_\lambda/\lambda^2$ then R is a closed Hecke operator. For this operator we have $\mathrm{Tr}(B) = 0$, hence H_{R_T} is not co-semi-simple.

Using Theorem 4.2 we can also show the non-semi-simplicity of the quantum semi-groups $\mathrm{GL}_q(m|n)$ given in [23]. Details will be given elsewhere.

Appendix A. The Hecke Algebras

We recall in this Appendix the definition and some important properties of the Hecke algebra $\mathcal{H}_n$ proved by R. Dipper and G. James (Theorem A.1), the reader is referred to [1, 2] for details. Then we derive from them some results that are needed in our context (Lemmas A.2, A.3, A.4).

The symmetric groups $\mathfrak{S}_n$ consists of permutations of the set $\{1, 2, \ldots, n\}$. It is generated be the basic transpositions $v_i = (i, i+1), 1 \leq i \leq n-1$, acting on $\{1, 2, \ldots, n\}$ as follows:

$$\{1, 2, \ldots, n\}v_i = \{1, 2, \ldots, i-1, i+1, i, i+2, \ldots, n\}.$$

The $v_i's$ satisfy the following relations:

$$\begin{aligned} &v_i v_{i+1} v_i = v_{i+1} v_i v_{i+1}, \\ &v_i v_j = v_j v_i, |i-j| \geq 2, \\ &v_i^2 = 1. \end{aligned}$$

The length of a permutation v is the number of pairs $(i, j), 1 \leq i < j \leq n$, such that $iv > jv$. This is equal to the minimal number of v_i's needed to express v. This fact follows from the following property of the length:

$$l(v_i v) - l(v) = \begin{cases} 1 & \text{iff } iv < (i+1)v; \\ -1 & \text{otherwise.} \end{cases}$$

Irreducible representations of $\mathfrak{S}_n$ are indexed by partitions of n. A partition λ of n (written as $\lambda \vdash n$) is a sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots$, such that $\sum_i \lambda_i = n$. Any sequence of non-negative integers, the sum of which is equal to n is called a composition of n, (written as $\lambda \models n$). Given a composition $\lambda = (\lambda_1, \lambda_2, \cdots)$, define $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$, setting $\lambda'_i := \#\{\lambda_j | \lambda_j \geq i\}$, then λ' is a partition.

Let λ be a composition of n, a λ-tableau is a matrix $a_i^j, 1 \leq j \leq \lambda_i, i = 1, 2, \ldots$, where a_i^j are different elements from the set $\{1, 2, \ldots, n\}$. Let t^λ be the λ-tableau in which the number $1, 2, \ldots, n$ appear in oder successive rows. The symmetric group $\mathfrak{S}_n$ acts transitively on the set of λ-tableaux. Let denote by $\mathfrak{S}_\lambda$ the row stabilizer of t^λ. Thus, for $\lambda \models n$, $\mathfrak{S}_\lambda \cong \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots$. Let w_λ be the element of $\mathfrak{S}_n$, that takes t^λ to the λ-tableau which has $1, 2, \ldots, n$ in order down successive columns. A tableau is said to be row- (column-) standard if the number $1, 2, \ldots, n$ increase along the rows (columns) and standard if it is row- and column-standard.

The Hecke Algebra $\mathcal{H}_n$. Let k be a field of characteristic zero, $0 \neq q \in k$. As a k-vector space, the Hecke algebra $\mathcal{H}_{n,q}$ is spanned by $\{T_v | v \in \mathfrak{S}_n\}$. The multiplication satisfies the following relations:

$$\begin{aligned} &T_{\mathbf{1}} = \mathbf{1}_{\mathcal{H}_{n,q}}, \\ &T_v T_w = T_{vw} \text{ iff } l(wv) = l(v) + l(w), \\ &T_{v_i^2} = (q-1)T_{v_i} + q\mathbf{1},\ v_i = (i, i+1). \end{aligned}$$

Let denote $T_i := T_{v_i}$, $1 \leq i \leq n-1$. As is the case of $\mathfrak{S}_n$, T_i generate $\mathcal{H}_n$ and satisfy:

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, 1 \leq i \leq n-2.$$

For a composition λ of n, denote $\mathcal{H}_\lambda$ the subalgebra of $\mathcal{H}_{n,q}$, generated by $\{T_v | v \in \mathfrak{S}_\lambda\}$. Then, $\mathcal{H}_\lambda \cong \mathcal{H}_{\lambda_1} \times \mathcal{H}_{\lambda_2} \times \cdots$.

Semi-simplicity of $\mathcal{H}_n$. If q is not a root of unity then $\mathcal{H}_n = \mathcal{H}_{n,q}$ is semi-simple. The reader is referred to any text book on algebra for the general theory of semi-simple algebras, e.g., Van der Waerden [30]. Simple modules of $\mathcal{H}_n$ are indexed by partitions of n. In fact, let denote $x_\lambda := \sum_{w \in \mathfrak{S}_\lambda} T_w$, $y_\lambda := \sum_{w \in \mathfrak{S}_\lambda} (-q)^{-l(w)} T_w$, $z_\lambda := \sum_{w \in \mathfrak{S}_\lambda} x_\lambda T_{w_\lambda} y_{\lambda'}$, then the right ideal $S_\lambda := z_\lambda \mathcal{H}_n$

is a simple right $\mathcal{H}_n$-module. For example, for the partition $\lambda = (n)$, $\lambda' = (1, 1, \ldots, 1)$, we have: $z_{(n)} = \sum_{w\in\mathfrak{S}_n} T_w$, $z_{(1,1,\ldots,1)} := \sum_{w\in\mathfrak{S}_n} (-q)^{-l(w)} T_w$. Let denote, for $n \in \mathbb{Z}$,

$$[n]_q := \begin{cases} \frac{q^n-1}{q-1} & \text{if } n \neq 0; \\ 0 & \text{if } n = 0, \end{cases}$$

and $[\pm n]_q! := [\pm 1]_q [\pm 2]_q \cdots [\pm n]_q$, $[0]_q! := 1$. Let $x_n := \frac{1}{[n]_q!} z_{(n)}$, $y_n := \frac{1}{[n]_{1/q}!} z_{(1,1,\cdots,1)}$, then x_n and y_n are central:

$$T_w x_n = x_n T_w = q^{l(w)} x_n, \tag{60}$$

$$T_w y_n = y_n T_w = (-1)^{l(w)} y_n. \tag{61}$$

Hence x_n and y_n are idempotents and the ideals $S_{(n)}, S_{(1,1,\ldots,1)}$ are two sided and of dimension 1 over k. We also denote, for $m \leq n$:

$$x_m := \frac{1}{[m]_q!} z_{(m,1,\ldots,1)} = \frac{1}{[m]_q!} \sum_{w\in\mathfrak{S}_{(m,1,\ldots,1)}} T_w, \tag{62}$$

$$y_m := \frac{1}{[m]_{1/q}!} z_{(1,1,\cdots,1,n-m)} = \frac{1}{[m]_{1/q}!} \sum_{w\in\mathfrak{S}_{(1,1,\ldots,1,n-m)}} (-q)^{-l(w)} T_w. \tag{63}$$

Murphy Operators and Primitive Idempotents. Primitive idempotents of $\mathcal{H}_n$ can be expressed in term of Murphy operators. The Murphy operators L_m, $1 \leq m \leq n$, were introduced in [2] as follows: $L_1 := 0$[1],

$$L_m := q^{-1} T_{m-1} + q^{-2} T_{m-2} T_{m-1} T_{m-2} + \cdots + q^{-m+1} T_1 T_2 \cdots T_{m-1} \cdots T_1, m \geq 2.$$

The following equality will be frequently used:

$$T_k T_{k+1} \cdots T_{m-1} T_m T_{m-1} \cdots T_{k+1} T_k = T_m T_{m-1} \cdots T_{k+1} T_k T_{k+1} \cdots T_{m-1} T_m, \qquad (k < m \leq n-1).$$

We summarize some properties of Murphy operators obtained in [2] in a theorem.

Theorem A.1. (i) *L_m, $1 \leq m \leq n$ commute and symmetric polynomials on them form the center of $\mathcal{H}_n$ ([2], Theorem 2.14).*

(ii) *For each simple module S_λ, $\lambda \vdash n$, there exists a basis (called Young's semi-normal basis) $l_1, l_2, \ldots, l_d$, $d = \dim_k(S_\lambda)$, such that*

$$l_i L_m = [r_{i,\lambda}(m)]_q l_i, \tag{64}$$

$r_{i,\lambda}(m) := k - j$, where (j, k) is the coordinate of m in certain standard λ-tableau ([2], Lemma 4.6). [2]

(iii) *The set*

$$\left\{ e_{i,\lambda} := \prod_{|k|\leq n-1} \prod_{\substack{1\leq m\leq n \\ k\neq r_{i,\lambda}(m)}} \frac{L_m - [k]_q}{[r_{i,\lambda}(m)]_q - [k]_q} \,\middle|\, 1 \leq i \leq d, \lambda \vdash n \right\}$$

is a complete set of primitive idempotents of $\mathcal{H}_n$ ([2], Theorem 5.2).

The following lemma contains some direct consequences of Theorem.

[1] There was a misprint in [2] where $L_1 := 1$.
[2] In [2], $r_{i,\lambda}(m)$ is defined to be $[k-j]_q$.

Lemma A.2. (i) $|r_{i,\lambda}(m)| \leq m-1$ *and*

$$\prod_{|c|\leq m-1} (L_m - [c]_q) = 0. \tag{65}$$

(ii) *For a sequence $a = (a_2, a_3, \ldots, a_n)$ of integers, $|a_i| \leq m$, let denote*

$$e_a := \prod_{2\leq m\leq n} \prod_{\substack{|c|\leq m-1\\ c\neq a_m}} \frac{L_m - [c]_q}{[a_m]_q - [c]_q},$$

then the set $\{e_a | e_a \neq 0\}$ is a complete set of primitive idempotents in $\mathcal{H}_n$.

(iii)

$$\prod_{\substack{|c|\leq n-1\\ c\neq n-1}} \frac{(L_n - [c]_q)}{[n-1]_q - [c]_q} = x_n, \tag{66}$$

and

$$\prod_{\substack{|c|\leq n-1\\ c\neq -n+1}} \frac{(L_n - [c]_q)}{[-n+1]_q - [c]_q} = y_n, \tag{67}$$

(iv)

$$x_n = \prod_{m=2}^{n} \frac{L_m - [-1]_q}{[m-1]_q - [-1]_q}, \tag{68}$$

$$y_n = \prod_{m=2}^{n} \frac{L_m - [1]_q}{[-m+1]_q - [1]_q}. \tag{69}$$

Proof. Remark that by its definition, the coefficients $r_{i,\lambda}(n)$ satisfy:

$$|r_{i,\lambda}(n)| = |j-k| \leq n-1.$$

The equation hold only when either $\lambda = (n)$ or $\lambda = (1, 1, \ldots, 1)$. Hence

$$\prod_{|c|\leq n-1} (L_n - [c]_q) = 0.$$

Since, for $m < n$, the operator L_m is an element of $\mathcal{H}_{(m,1,\ldots,1)} \cong \mathcal{H}_m$, we have

$$\prod_{|c|\leq m-1} (L_m - [c]_q) = 0.$$

From this equation and (64), one derives

$$|r_{i,\lambda}(m)| \leq m-1.$$

Thus we proved (i).

To prove (ii) we remark that, by virtue of Part (i), the following statements hold:

(1) E_a are idempotents and mutual orthogonal, their sum is $\mathbf{1}$.
(2) $E_{i,\lambda} = E_{(r_{i,\lambda}(2), r_{i,\lambda}(3), \ldots, r_{i,\lambda}(n))}$, i.e., $\{E_{i,\lambda} | \lambda \vdash n, 1 \leq i \leq \dim_k S_\lambda\}$ is a subset of $\{E_a | a \in A\}$.

Since the set $\{E_{i,\lambda}|\lambda \vdash n, 1 \leq i \leq \dim_k S_\lambda\}$ is a complete set of primitive idempotents, the assertion follows.

To a proof of (iii) we remark that the element

$$x = \prod_{\substack{|c|\leq n-1\\ c\neq n-1}} \frac{L_n}{[n-1]_q - [c]_q}$$

vanishes on all simple modules except on $S_{(n)} = x_n\mathcal{H}_n$. Hence, $x \in x_n\mathcal{H}_n$. According to (60), $xx_n = x_n$, whence $x = x_n$. The second statement can be proved analogously.

(iv) can be proved easily by induction. The assertion holds for $n = 2$ evidently. Further we have, by virtue of (60):

$$\prod_{m=2}^{n+1} \frac{L_m - [-1]_q}{[m-1]_q - [-1]_q} = \frac{x_n(q^{-1}L_{n+1} + 1)}{[n+1]_q}$$

$$= \frac{1}{[n+1]_q} x_n(1 + T_n + \cdots + T_nT_{n-1}\cdots T_1) = x_{n+1}.$$

The assertion for y_n can be proved quite analogously. ∎

Lemma A.3. *Let R be an even Hecke symmetry of rank r and $\rho_n : H_n \longrightarrow \overline{\mathcal{H}}_n$ be the corresponding representation of $\mathcal{H}_n$. Then $(\overline{L_n} - [-r]_q)$ is invertible in $\overline{\mathcal{H}}_n$.*

Proof. $\overline{\mathcal{H}}_n$ is semi-simple and is isomorphic to a two sided ideal of $\mathcal{H}_n$. Hence any simple module of $\mathcal{H}_n$ is either simple module of $\overline{\mathcal{H}}_n$ or zero. By the assumption $\overline{y}_\lambda = 0$ whenever $\lambda_{r+1} \neq 0$, where $\lambda \vdash n$. Hence a simple module S_λ of $\mathcal{H}_n$ remains simple $\overline{\mathcal{H}}_n$-module only if $\lambda_{r+1} = 0$. For these λ, $r_{i,\lambda}(m) \geq -r+1$. Hence, according to Theorem A.1, (ii), $\overline{L_n} - [-r]_q$ is invertible on $\mathcal{S}_\lambda$. Thus, $\overline{L_n} - [-r]_q$ is invertible on $\overline{\mathcal{H}}_n$.∎

The completion of Proposition 3.2. Let $\overline{\mathcal{H}_n}$ be a factor algebra of $\mathcal{H}_n$ by a two-sided ideal $\mathcal{I}$. Then $\overline{\mathcal{H}_n}$ is isomorphic as an algebra to a two-sided ideal of $\mathcal{H}_n$, namely, the ideal $\mathcal{I}^\perp := \{h \in \mathcal{H}_n | h\mathcal{I} = 0\}$. $\mathcal{H}_n$ is then isomorphic as an algebra to the direct product of $\overline{\mathcal{H}_n}$ and $\mathcal{I}$. Assume that $n = r+m$, $r, p \geq 1$, and y_{r+1}, defined as in (63), belongs to $\mathcal{I}$. For an element $h \in \mathcal{H}_n$ denote $\overline{h}$ its class in $\overline{\mathcal{H}_n}$.

Lemma A.4. *Let $\theta \in \mathcal{H}_{(1,1,\ldots,1,m)} \cong \mathcal{H}_r$ be a primitive idempotent in $\mathcal{H}_{(1,1,\ldots,1,m)}$. Then $\overline{y}_n\theta$ is a primitive idempotent in $\overline{\mathcal{H}}_n$.*

Proof. Let denote L^r_k the image of L_k under the embedding $\mathcal{H}_m \ni T_j \longrightarrow T_{r+j} \in \mathcal{H}_n$. According to Theorem A.1, assume that

$$\theta = \prod_{\substack{|k|\leq m\\ |c|\leq k-1}} \frac{\overline{L}^r_k - [c]_q}{[r_{i,\lambda}(k)]_q - [c]_q},$$

where λ is a partition of m. According to (69), $\overline{y}_r(\overline{1} - \overline{L}_{r+1}) = [r+1]_q\overline{y}_{r+1} = 0$. On the other hand, for $k \geq 1$,

$$L_{r+k} = L^r_k + q^{-k+1}T_{r+k-1}\cdots T_{r+1}L_{r+1}T_{r+1}\cdots T_{r+k-1}.$$

Hence

$$\overline{y}_r\overline{L}_{r+k} = \overline{y}_r\overline{L}^r_k + q^{-k+1}\overline{T}_{r+k-1}\cdots\overline{T}_{r+1}\overline{y}_r\overline{L}_{r+1}\overline{T}_{r+1}\cdots\overline{T}_{r+k-1} = \overline{y}_r(1 + q\overline{L}^r_k).$$

Therefore

$$\overline{y}_r(\overline{L}_{r+k} - [c]_q) = \overline{y}_r(1 + q\overline{L}^r_k - [c]_q) = q\overline{y}_r(\overline{L}^r_k - [c-1]_q). \tag{70}$$

Thus

$$\overline{y}_r \prod_{\substack{1\le k\le m \\ |c|\le k-1 \\ c\ne r_{i,\lambda}(k)}} \frac{q^{-1}(\overline{L}_{r+k} - [c+1]_q)}{[r_{i,\lambda(k)}]_q - [c]_q} = \overline{y_r\theta}.$$

Equations (70) and (65) imply

$$\overline{y}_r \prod_{|c|\le k-1} (\overline{L}_{r+k} - [c+1]_q) = 0.$$

Hence, for $m \ge 1$,

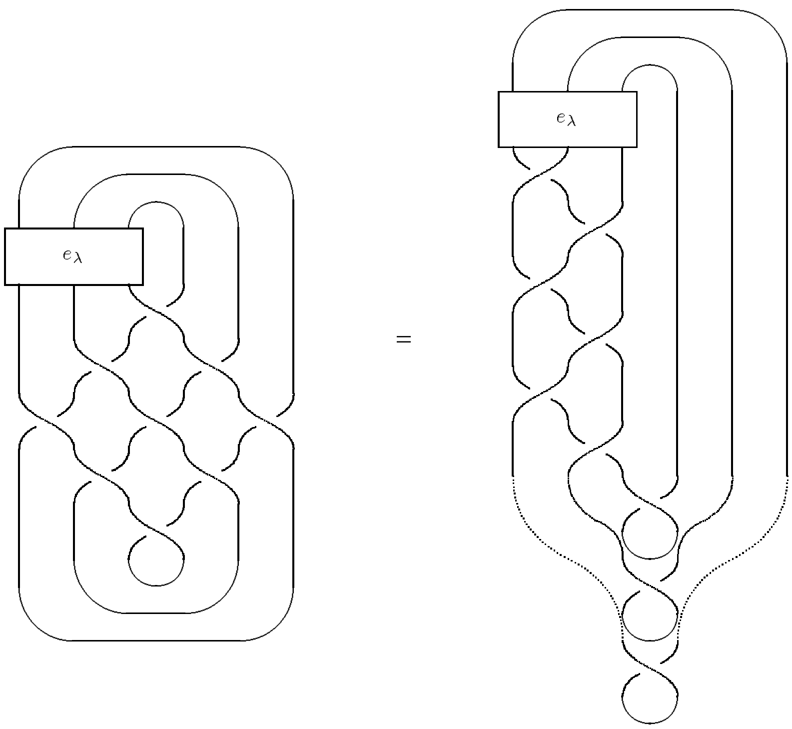
$$\overline{y}_r \prod_{\substack{|c+1|\le r+k-1 \\ c\ne r_{i,\lambda}(k)}} \frac{\overline{L}_{r+k} - [c+1]_q}{[r_{i,\lambda(k)}+1]_q - [c+1]_q} = \overline{y}_r \prod_{\substack{|c+1|\le k-1 \\ c\ne r_{i,\lambda}(k)}} \frac{\overline{L}_{r+k} - [c+1]_q}{[r_{i,\lambda(k)}+1]_q - [c+1]_q}.$$

Since y_r is an idempotent, we have:

$$\overline{y}_r \prod_{\substack{1\le k\le m \\ |c+1|\le r+k-1 \\ c\ne r_{i,\lambda}(k)}} \frac{\overline{L}_{r+k} - [c+1]_q}{[r_{i,\lambda(k)}+1]_q - [c+1]_q} = \overline{y}_r\theta.$$

The left-hand side term of this equality is a primitive idempotent, hence, so is the right-hand side term.∎

Appendix to Equation (40).



Proof of Proposition 3.11. Let denote, for $n, k \in \mathbb{Z}$

$$\binom{[n]_q}{[k]_q} := \begin{cases} \frac{[n]_q!}{[k]_q![n-k]_q!} & \text{if } n \geq k \geq 0 \\ 0 & \text{otherwise .} \end{cases}$$

We have, for all $m, n \in \mathbb{Z}$,

$$\begin{aligned} [m+n]_q &= [m]_q + q^m [n]_q \\ &= [n]_q + q^n [m]_q. \end{aligned}$$

Hence

$$\binom{[n]_q}{[k]_q} = \binom{[n-1]_q}{[k]_q} + q^{n-k} \binom{[n-1]_q}{[k-1]_q},$$

for all $m, n \in \mathbb{Z}$. These equations will be frequently used below. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. Setting $a_i = \lambda_i + r - i$, we have

$$\begin{aligned} \mathrm{rdim}(M_\lambda) &= \det\left(q^{-r(\lambda_i - i + j)} \binom{[r+\lambda_i - i + j - 1]_q}{[r-1]_q} \right)_{1 \leq i,j \leq r} \\ &= q^{-r\sum_1^r \lambda_i} \det\left(\binom{[a_i + j - 1]_q}{[r-1]_q} \right)_{1 \leq i,j \leq r} = (*) \end{aligned}$$

Let us compute the last determinant. In what follows we shall use $[n]$ instead of $[n]_q$ for convenience.

$$(*) = q^{-r\sum_1^r \lambda_i} \begin{vmatrix} \binom{[a_1]}{[r-1]} & \binom{[a_1+1]}{[r-1]} & \cdots & \binom{[a_1+r-2]}{[r-1]} & \binom{[a_1+r-1]}{[r-1]} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{[a_r]}{[r-1]} & \binom{[a_r+1]}{[r-1]} & \cdots & \binom{[a_r+r-2]}{[r-1]} & \binom{[a_r+r-1]}{[r-1]} \end{vmatrix}$$

(subtract the r-th column by the $(r-1)$-th column and then the $(r-1)$-th column by the $(r-2)$-th column, and so on...)

$$= q^{-r\sum_1^r \lambda_i} \begin{vmatrix} \binom{[a_1]}{[r-1]} & q^{a_1-r+2}\binom{[a_1]}{[r-2]} & \cdots & q^{a_1-1}\binom{[a_1+r-3]}{[r-2]} & q^{a_1}\binom{[a_1+r-2]}{[r-2]} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{[a_r]}{[r-1]} & q^{a_r-r+2}\binom{[a_r]}{[r-2]} & \cdots & q^{a_r-1}\binom{[a_r+r-3]}{[r-2]} & q^{a_r}\binom{[a_r+r-2]}{[r-2]} \end{vmatrix}$$

$$= q^{-\frac{(r-1)(r-2)}{2} - r\sum_1^r \lambda_i} \begin{vmatrix} \binom{[a_1]}{[r-1]} & q^{a_1}\binom{[a_1]}{[r-2]} & \cdots & q^{a_1}\binom{[a_1+r-3]}{[r-2]} & q^{a_1}\binom{[a_1+r-2]}{[r-2]} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{[a_r]}{[r-1]} & q^{a_r}\binom{[a_r]}{[r-2]} & \cdots & q^{a_r}\binom{[a_r+r-3]}{[r-2]} & q^{a_r}\binom{[a_r+r-2]}{[r-2]} \end{vmatrix}$$

(repeat the above process, this time only for $r-2$ last columns)

$$= q^{-\frac{(r-1)(r-2)}{2} - r\sum_1^r \lambda_i} \times$$
$$\begin{vmatrix} \binom{[a_1]}{[r-1]} & q^{a_1}\binom{[a_1]}{[r-2]} & \cdots & q^{2a_1-1}\binom{[a_1+r-3]}{[r-2]} & q^{2a_1}\binom{[a_1+r-2]}{[r-2]} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{[a_r]}{[r-1]} & q^{a_r}\binom{[a_r]}{[r-2]} & \cdots & q^{2a_r-1}\binom{[a_r+r-3]}{[r-2]} & q^{2a_r}\binom{[a_r+r-2]}{[r-2]} \end{vmatrix}$$

$$= \cdots\cdots =$$

$$= q^{-\sum_1^r \frac{(i-1)(i-2)}{2} - r\sum_1^r \lambda_i} \times \begin{vmatrix} \binom{[a_1]}{[r-1]} & q^{a_1}\binom{[a_1]}{[r-2]} & \cdots & q^{(r-2)a_1}\binom{[a_1+r-3]}{[r-2]} & q^{(r-1)a_1}\binom{[a_1+r-2]}{[r-2]} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{[a_r]}{[r-1]} & q^{a_r-r+2}\binom{[a_r]}{[r-2]} & \cdots & q^{(r-2)a_r}\binom{[a_r+r-3]}{[r-2]} & q^{(r-1)a_r}\binom{[a_r+r-2]}{[r-2]} \end{vmatrix}$$

$$= \frac{q^{(r-1)(\sum_1^r a_i) - \sum_1^r \frac{(i-1)(i-2)}{2} - r\sum_1^r \lambda_i}}{\prod_1^r [r-i]_!} \times \begin{vmatrix} q^{-(r-1)a_1}\prod_0^{r-2}[a_1-i] & \prod_0^{r-3}[a_1-i] & \cdots & q^{-a_1}[a_1]_q & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q^{-(r-1)a_r}\prod_0^{r-2}[a_r-i] & \prod_0^{r-3}[a_r-i] & \cdots & q^{-a_r}[a_1]_q & 1 \end{vmatrix}$$

$$= \frac{q^{(r-1)(\sum_1^r a_i) - \sum_1^r \frac{(i-1)(i-2)}{2} - r\sum_1^r \lambda_i}}{\prod_1^r [r-i]_!} \times \begin{vmatrix} q^{-(r-1)a_1}\prod_0^{r-2}([a_1]-q^{a_1-i}[i]) & \prod_0^{r-3}([a_1]-q^{a_1-i}[i]) & \cdots & q^{-a_1}[a_1]_q & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q^{-(r-1)a_r}\prod_0^{r-2}([a_r]-q^{a_r-i}[i]) & \prod_0^{r-3}([a_r]-q^{a_r-i}[i]) & \cdots & q^{-a_r}[a_r]_q & 1 \end{vmatrix}$$

$$= \frac{q^{(r-1)(\sum_1^r a_i) - \sum_1^r \frac{(i-1)(i-2)}{2} - r\sum_1^r \lambda_i}}{\prod_1^r [r-i]_!} \times \begin{vmatrix} \prod_0^{r-2}(q^{-a_1}[a_1]-q^{-i}[i]) & \prod_0^{r-3}(q^{-a_1}[a_1]-q^{-i}[i]) & \cdots & q^{-a_1}[a_1]_q & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \prod_0^{r-2}(q^{-a_r}[a_r]-q^{-i}[i]) & \prod_0^{r-3}(q^{-a_r}[a_r]-q^{-i}[i]) & \cdots & q^{-a_r}[a_r]_q & 1 \end{vmatrix}$$

$$= \frac{q^{(r-1)(\sum_1^r a_i) - \sum_1^r \frac{(i-1)(i-2)}{2} - r\sum_1^r \lambda_i}}{\prod_1^r [r-i]_!} \times \begin{vmatrix} (q^{-a_1}[a_1])^{r-1} & (q^{-a_1}[a_1])^{r-2} & \cdots & q^{-a_1}[a_1]_q & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (q^{-a_r}[a_r])^{r-1} & (q^{-a_r}[a_r])^{r-2} & \cdots & q^{-a_r}[a_r]_q & 1 \end{vmatrix}$$

$$= q^{(r-1)(\sum_1^r a_i) - \sum_1^r \frac{(i-1)(i-2)}{2} - r\sum_1^r \lambda_i} \prod_{1\le i<j\le r} \frac{q^{-a_i}[a_i]-q^{-a_j}[a_j]}{[j-i]}$$

$$= q^{(r-1)(\sum_1^r a_i) - \sum_1^r \frac{(i-1)(i-2)}{2} - r\sum_1^r \lambda_i - \sum_1^r (r-i)a_i} \prod_{1\le i<j\le r} \frac{[a_i-a_j]}{[j-i]}$$

$$= q^{-\sum_1^r \lambda_i(r+1-i)} \prod_{1\le i<j\le r} \frac{[\lambda_i-\lambda_j+j-i]}{[j-i]}$$

∎

References

[1] R. Dipper and G. James. Representations of Hecke Algebras of General Linear Groups. *Proc. London Math. Soc.*, 52(3):20–52, 1986.

[2] R. Dipper and G. James. Block and Idempotents of Hecke Algebras of General Linear Groups. *Proc. London Math. Soc.*, 54(3):57–82, 1987.

[3] J. Du, B. Parshall, and J. Wang. Two-parameter Quantum Linear Groups and the Hyperbolic Invariance or q-Schur Algebras. *Journal of London Math. Soc.*, 44(2):420–436, 1991.
[4] L. Faddeev, N. Reshetikhin, and L. Takhtajian. Quantisation of Lie Groups and Lie Algebras. *Leningrad Math. Journal*, 1:193–225, 1990.
[5] J.A. Green. Locally Finite Representations. *Journal of Algebra*, 41:137–171, 1976.
[6] D. Gurevich. Hecke Symmetry and Quantum Determinants. *Soviet Math. Doklad*, 38:555–559, 1989.
[7] D.I. Gurevich. Algebraic Aspects of the Quantum Yang-Baxter Equation. *Leningrad Math. Journal*, 2(4):801–828, 1991.
[8] D.I. Gurevich, P.N. Pyatov, and P.A. Saponov. Hecke Symmetries and Characteristic Relations on Reflection Equation Algebras. *q-alg*, 9605048, 1996.
[9] Phung Ho Hai. Hopf Envelope of A Rigid Coquasitriangular Bialgebra. unpublished, 1994.
[10] Phung Ho Hai. Koszul Property and Poincaré Series of Matrix Bialgebra of Type A_n. *Journal of Algebra*, 192(2):734–748, 1997.
[11] T. Hayashi. Quantum Groups and Quantum Determinant. *Publ. RIMS, Kyoto Univ.*, 28:57–81, 1992.
[12] T. Hayashi. Quantum Groups and Quantum Semigroups. *preprint*, 1996.
[13] G.D. James. *The Representation Theory of Symmetric Groups.* Lecture Note in Mathematics. Springer Verlang, Berlin, 1978.
[14] A. Joyal and R. Street. The Geometry of Tensor Calculus. *Advances in Math.*, 88:55–112, 1991. MR 92d:18011.
[15] A. Joyal and R. Street. Tortitle Yang-Baxter Operators in Tensor Categories. *J. Pure Appl. Algebra*, pages 43–51, 1991. MR 92e:18006.
[16] C. Kassel. *Quantum Groups.* Springer-Verlag, 1995.
[17] R. Larson and J. Towber. Two Dual Classes of Bialgebras Related To The Concepts of "Quantum Groups" and "Quantum Lie Algebra". *Comm. in Algebra*, 19(12):3295–3345, 1991.
[18] V. Lyubashenko. Superanalysis and Solutions to the Triangles Equation. *Ph.D Dissertation Phy. - Mat. Sciences, Kiev, (in Russian)*, 1987.
[19] V.V. Lyubashenko. Hopf Algebras and Vector Symmetries. *Russian Math. Survey*, 41(5):153–154, 1986.
[20] V.V. Lyubashenko and A. Sudbery. Quantum Super Groups of $\mathrm{GL}(n|m)$ Type: Differential Forms, Koszul Complexes and Berezinians. *Hep-th*, 1993.
[21] S. Majid. Quasitriangular Hopf Algebras and Yang-Baxter Equation. *International Journal of Modern Physics A*, 5:1–91, 1990.
[22] Yu.I. Manin. *Quantum Groups and Non-commutative Geometry.* GRM, Univ. de Montreal, 1988.
[23] Yu.I. Manin. Multiparametric Quantum Deformation of the General Linear Supergroups. *Com. Math. Phys.*, 123:163–175, 1989.
[24] P.N. Pyatov and P.A. Saponov. Newton Relations for Quantum Matrix Algebras of RTT-Type. *Preprint IHEP*, 1996.
[25] P. Schauenburg. On Coquasitriangular Hopf Algebras and the Quantum Yang-Baxter Equation. *Algebra Berichte 67, Verlag Reinhard Fischer, Munich*, 1992.
[26] M. Sweedler. *Hopf Algebras.* Benjamin, New York, 1969.
[27] M. Takeuchi. Free Hopf Algebra Generated by Coalgebras. *Journal Math. Soc. Japan*, 23:561–582, 1971.
[28] M. Takeuchi and D. Tambara. A new one-parameter family of 2×2 quantum matrices. *Proc. Japan Acad.*, 8(8), 1991.
[29] V.G. Turaev. Modular Categories and 3-Manifolds Invariant. *Intern. J. Modern Phys. B*, 6:1807–1824, 1992.
[30] B.L. van der Waerden. *Algebra.* Springer-Verlag, 1967.

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