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Notes On Stable Maps And Quantum Cohomology

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Dedicated to the memory of Claude Itzykson

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0. Introduction

0.1. Overview. The aim of these notes is to describe an exciting chapter in the recent development of quantum cohomology. Guided by ideas from physics (see [W]), a remarkable structure on the solutions of certain rational enumerative geometry problems has been found: the solutions are coefficients in the multiplication table of a quantum cohomology ring. Associativity of the ring yields non-trivial relations among the enumerative solutions. In many cases, these relations suffice to solve the enumerative problem. For example, let N_d be the number of degree d, rational plane curves passing through $3d-1$ general points in $\mathbf{P}^2$. Since there is a unique line passing through 2 points, $N_1 = 1$. The quantum cohomology ring of $\mathbf{P}^2$ yields the following beautiful associativity relation determining all N_d for $d \geq 2$:

$$N_d = \sum_{d_1+d_2=d,\ d_1,d_2>0} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

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Similar enumerative formulas are valid on other homogeneous varieties. Viewed from classical enumerative geometry, the quantum ring structure is a complete surprise.

The path to quantum cohomology presented here follows the work of Kontsevich and Manin. The approach is algebro-geometric and involves the construction and geometry of a natural compactification of the moduli space of maps. The large and exciting conjectural parts of the subject of quantum cohomology are avoided here. We focus on a part of the story where the proofs are complete. We also make many assumptions that are not strictly necessary, but which simplify the presentation.

It should be emphasized that this is in no way a survey of quantum cohomology, or any attempt at evaluating various approaches. In particular, the symplectic point of view is not covered (see [R-T]). These notes are based on a jointly taught course at the University of Chicago in which our main efforts were aimed at understanding the papers of Kontsevich and Manin. We thank R. Donagi for instigating this course. Thanks are due to D. Abramovich, P. Belorousski, H. Tamvakis, and A. Vistoli for comments and suggestions. Another algebro-geometric approach, using a different compactification, can be found in [L-T 1].

0.2. Notation. In this exposition, for simplicity, we consider only homology classes of even dimension. To avoid doubling indices, we set, for a complete variety X,

$$A_d X = H_{2d}(X, \mathbb{Z}), \quad A^d X = H^{2d}(X, \mathbb{Z}).$$

When X is nonsingular of dimension n, identify $A^d X$ with $A_{n-d} X$ by the Poincaré duality isomorphism

$$A^d X \stackrel{\sim}{\to} A_{n-d} X, \quad c \mapsto c \cap [X].$$

In particular, a closed subvariety V of X of pure codimension d determines classes in $A_{n-d} X$ and $A^d X$ via the duality isomorphism. Both of these classes are denoted by $[V]$. For homogeneous varieties, which are our main concern, the Chow groups coincide with the topological groups. Hence $A_d X$ and $A^d X$ can be taken to be the Chow homology and cohomology groups for homogeneous varieties (see [F]).

If X is complete, and c is a class in the ring $A^* X = \bigoplus A^d X$, and β is a class in $A_k X$, we denote by $\int_\beta c$ the degree of the class of the zero cycle obtained by evaluating c_k on β, where c_k is the component of c in $A^k X$. When V is a closed, pure dimensional subvariety of X, we write $\int_V c$ instead of $\int_{[V]} c$. We use cup product notation $\cup$ for the product in $A^* X$.

We will be concerned only with varieties over $\mathbb{C}$ since the relevant moduli spaces have not yet been constructed in positive characteristic. Let $[n]$ denote the finite set of integers $\{1, 2, \ldots, n\}$.

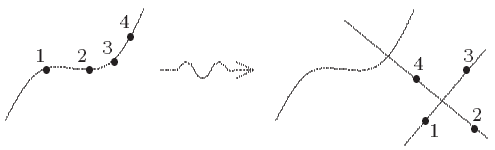
0.3. Compactifications of moduli spaces. An important feature of quantum cohomology is the use of intersection theory in a space of maps of curves into a variety, rather than in the variety itself. To carry this out, a good compactification of such a space is required. At least when X is sufficiently positive, Kontsevich has constructed such a compactification. We start, in this section, by reviewing some related moduli spaces with similar properties. Kontsevich's space of stable maps will be introduced in section 0.4.

Algebraic geometers by now have become quite comfortable working with the moduli space M_g of projective nonsingular curves of genus g, and its compactification $\overline{M}_g$, whose points correspond to projective, connected, nodal curves of

arithmetic genus g, satisfying a stability condition (due to Deligne and Mumford) that guarantees the curve has only a finite automorphism group. These moduli spaces are irreducible varieties of dimension $3g-3$ if $g \geq 2$, smooth if regarded as (Deligne-Mumford) stacks, and with orbifold singularities if regarded as ordinary coarse moduli spaces.

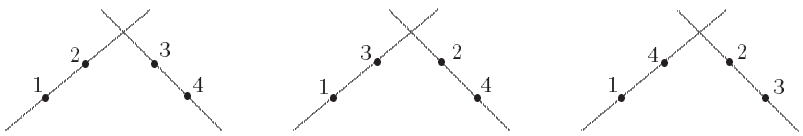
Some related spaces have become increasingly important. The moduli space $M_{g,n}$ parametrizes projective nonsingular curves C of genus g together with n distinct marked points $p_1, \ldots, p_n$ on C. $M_{g,n}$ has a compactification $\overline{M}_{g,n}$ whose points correspond to projective, connected, nodal curves C, together with n distinct, nonsingular, marked points, again with a stability condition equivalent to the finiteness of automorphism groups. $\overline{M}_{g,1}$ is the universal curve over $\overline{M}_g$.

The first remarkable feature of the space $\overline{M}_{g,n}$ is that it compactifies $M_{g,n}$ without ever allowing the points to come together. When points on a smooth curve approach each other, in fact, the curve sprouts off one or more components, each isomorphic to the projective line, and the points distribute themselves at smooth points on these new components, in a way that reflects the relative rates of approach.

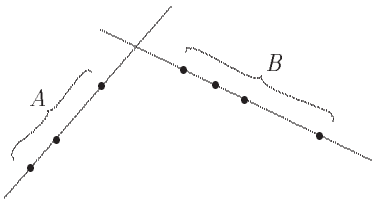


The spaces $\overline{M}_{g,n}$ again are smooth stacks, or orbifold coarse moduli spaces, of dimension $3g-3+n$, as long as this integer is nonnegative. The case of genus zero plays a prominent role in our story. In this case, $\overline{M}_{0,n}$ is a fine moduli space and a nonsingular variety. A point in $\overline{M}_{0,n}$ corresponds to a curve which is a tree of projective lines meeting transversally, with n distinct, nonsingular, marked points; the stability condition is that each component must have at least three special points, which are either the marked points or the nodes where the component meets the other components.

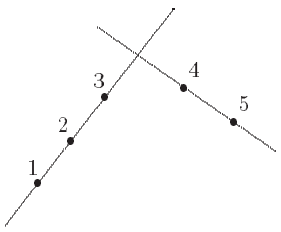
For $n = 3$, of course, $M_{0,3} = \overline{M}_{0,3}$ is a point. For $n = 4$, $M_{0,4}$ parametrizes 4 distinct marked points on a projective line. Since, up to isomorphism, one can fix the first three of these points, say to be 0, 1, and ∞, $M_{0,4}$ is isomorphic to $\mathbf{P}^1 \setminus \{0, 1, \infty\}$. It is not hard to guess what $\overline{M}_{0,4}$ must be. In fact, the three added points are represented by the following three marked curves:



In general, the closures of the loci of trees of a given combinatorial type are smooth subvarieties of $\overline{M}_{0,n}$, and all such loci meet transversally. There is a divisor $D(A|B)$ in $\overline{M}_{0,n}$ for each partition of $[n]$ into two disjoint sets A and B, each with at least two elements. A generic point of $D(A|B)$ is represented by two lines meeting transversally, with points labeled by A and B on each:



It is convenient to allow labeling by finite sets other than $[n]$; we write $\overline{M}_{g,A}$ for the corresponding moduli space where A is the labeling set. Let $B \subset A$ (if $g = 0$, then let $|B| \geq 3$). It is a fundamental fact that there is a morphism $\overline{M}_{g,A} \to \overline{M}_{g,B}$ which "forgets" the points marked in $A \setminus B$. On the open locus $M_{g,n}$ this map is the obvious one, but it is more subtle on the boundary: removing some points may make a component unstable, and such a component must be collapsed. For example, the map from $\overline{M}_{0,5}$ to $\overline{M}_{0,4}$ forgetting the point labeled 5 sends



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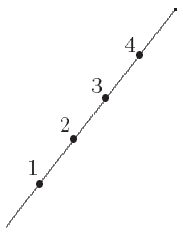

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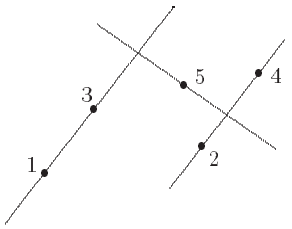

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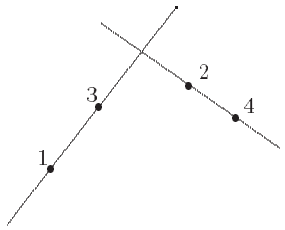
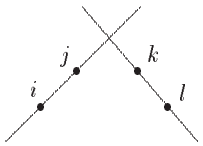


The algebra that shows this is a morphism is carried out in [Kn].

In particular, for any subset $\{i, j, k, l\}$ of four integers in $[n]$, we have a morphism from $\overline{M}_{0,n}$ to $\overline{M}_{0,\{i,j,k,l\}}$. The inverse image of the point $P(i, j \mid k, l)$



is a divisor on $\overline{M}_{0,n}$. This inverse image is a multiplicity-free sum of divisors $D(A|B)$: the sum is taken over all partitions $A \cup B = [n]$ satisfying $i, j \in A$ and $k, l \in B$. The fact that the three boundary points in $\overline{M}_{0,\{i,j,k,l\}} \cong \mathbf{P}^1$ are linearly equivalent implies their inverse images in $\overline{M}_{0,n}$ are linearly equivalent as well. Hence, the fundamental relation is obtained:

$$\sum_{i,j \in A \ k,l \in B} D(A|B) \ = \ \sum_{i,k \in A \ j,l \in B} D(A|B) \ = \ \sum_{i,l \in A \ j,k \in B} D(A|B) \tag{1}$$

in $A^1(\overline{M}_{0,n})$. Keel [Ke] has shown that the classes of these divisors $D(A|B)$ generate the Chow ring, and that the relations (1), together with the (geometrically obvious)

relations $D(A|B)\cdot D(A'|B') = 0$ if there are no inclusions among the sets A, B, A', B', give a complete set of relations.

0.4. The space of stable maps. In the remainder of the introduction, the basic ideas and constructions in quantum cohomology are introduced. The goal here is to give a precise overview with no proofs. The ideas introduced here are covered carefully (with proofs) in the main sections of these notes.

Let X be a smooth projective variety, and let β be an element in A_1X. Let $M_{g,n}(X,\beta)$ be the set of isomorphism classes of pointed maps $(C, p_1, \ldots, p_n, \mu)$ where C is a projective nonsingular curve of genus g, the markings $p_1, \ldots, p_n$ are distinct points of C, and μ is a morphism from C to X satisfying $\mu_*([C]) = \beta$. $(C, p_1, \ldots, p_n, \mu)$ is *isomorphic* to $(C', p_1', \ldots, p_n', \mu')$ if there is an scheme isomorphism $\tau : C \to C'$ taking p_i to p_i', with $\mu' \circ \tau = \mu$. Of course, if $\beta \neq 0$, $M_{g,n}(X,\beta)$ is empty unless β is the class of a curve in X. There are also other problems. For example, if $g = 0$, which will be the case of interest to us, $M_{g,n}(X,\beta)$ is empty if $\beta \neq 0$ and X contains no rational curves. To obtain a well-behaved space, one needs to make strong assumptions on X. In general, there is a compactification

$$M_{g,n}(X,\beta) \subset \overline{M}_{g,n}(X,\beta),$$

whose points correspond to stable maps $(C, p_1, \ldots, p_n, \mu)$ where C a projective, connected, nodal curve of arithmetic genus g, the markings $p_1, \ldots, p_n$ are distinct nonsingular points of C, and μ is a morphism from C such that $\mu_*([C]) = \beta$. Again, the stability condition (due to Kontsevich) is equivalent to finiteness of automorphisms of the map. Alternatively, $(C, p_1, \ldots, p_n, \mu)$ is a *stable* map if both of the following conditions hold for every irreducible component $E \subset C$:

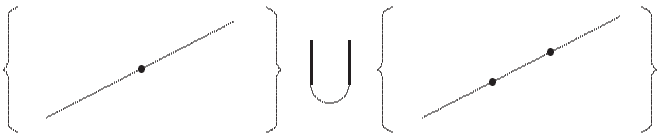
(1) If $E \cong \mathbf{P}^1$ and E is mapped to a point by μ, then E must contain at least three special points (either marked points or points where E meets the other components of C).
(2) If E has arithmetic genus 1 and E is mapped to a point by μ, then E must contain at least one special point.

Condition (2) is relevant only in case $g = 1$, $n = 0$, and $\beta = 0$ (in other cases, (2) is automatically satisfied). From conditions (1) and (2), it follows that $\overline{M}_{1,0}(X,0) = \emptyset$. Thus, in practice, (1) is the important condition.

When X is a point, so $\beta = 0$, one recovers the pointed moduli space of curves $\overline{M}_{g,n} \cong \overline{M}_{g,n}(\text{point}, 0)$. When $X \cong \mathbf{P}^r$ is a projective space, we write $\overline{M}_{g,n}(\mathbf{P}^r, d)$ in place of $\overline{M}_{g,n}(\mathbf{P}^r, d[\text{line}])$.

The simplest example is $\overline{M}_{0,0}(\mathbf{P}^r, 1)$, which is the Grassmannian $\mathbf{G}(\mathbf{P}^1, \mathbf{P}^r)$. If $n \geq 1$, $\overline{M}_{0,n}(\mathbf{P}^r, 1)$ is a locally trivial fibration over $\mathbf{G}(\mathbf{P}^1, \mathbf{P}^r)$ with the configuration space $\mathbf{P}^1[n]$ of [F-M] as the fiber. Let us look at the space $\overline{M}_{0,0}(\mathbf{P}^2, 2)$. An open set in this space is the space of nonsingular conics, since to each such conic D there is an isomorphism $\mathbf{P}^1 \xrightarrow{\sim} D \subset \mathbf{P}^2$, unique up to equivalence. Singular conics D that are the unions of two lines are similarly the isomorphic image $C \xrightarrow{\sim} D \subset \mathbf{P}^2$, where C is the union of two projective lines meeting transversally at a point. This gives:

We also have maps from the same C to $\mathbf{P}^2$ sending each line in the domain onto the same line in $\mathbf{P}^2$. To determine this map up to isomorphism, however, the point that is the image of the intersection of the two lines must be specified, so the data for this is a line in $\mathbf{P}^2$ together with a point on it. Finally, there are maps from $\mathbf{P}^1$ to a line in the plane that are branched coverings of degree two onto a line in the plane. These are determined by specifying the line together with the two branch points. The added points consist of:



Thus, we recover the classical space of complete conics – but in quite a different realization from the usual one. The same discussion is valid when $\mathbf{P}^2$ is replaced by $\mathbf{P}^r$, but this time the space is not the classical space of complete conics in space. The classical space specifies a plane together with a complete conic contained in the plane; Kontsevich's space has blown down all the planes containing a given line.

A complete nonsingular variety X is called *convex* if, for every morphism $\mu : \mathbf{P}^1 \to X$,

$$H^1(\mathbf{P}^1, \mu^*(T_X)) = 0.$$

Convexity is a very restrictive condition on X. Examples of convex spaces include all homogeneous spaces G/P where G is a semisimple Lie group and P is a parabolic subgroup. Hence, projective spaces, Grassmannians, smooth quadrics, flag varieties, and products of such spaces are all convex. The genus 0 moduli space of stable maps is well-behaved in case X is convex. In this case, $\overline{M}_{0,n}(X,\beta)$ exists as a projective nonsingular stack or orbifold coarse moduli space, containing $M_{0,n}(X,\beta)$ as a dense open subset. When $\overline{M}_{0,n}(X,\beta)$ is nonempty, its dimension is given by

$$\dim \overline{M}_{0,n}(X,\beta) = \dim X + \int_\beta c_1(T_X) + n - 3.$$

Here, $c_1(T_X)$ is the first Chern class of the tangent bundle to X. We assume always that the right side of this equation is nonnegative. In the stack or orbifold sense, this is a compactification with normal crossing divisors. When X is projective space, $\overline{M}_{0,n}(X,d)$ is irreducible. These assertions are Theorems 1–3 in these notes and are established in sections 1–6.

We will also write $\overline{M}_{0,A}(X,\beta)$ when the index set is a set A instead of $[n]$. These varieties also have forgetful morphisms $\overline{M}_{0,A}(X,\beta) \to \overline{M}_{0,B}(X,\beta)$ when B is a subset of A. In addition, if $|A| \geq 3$, there are morphisms $\overline{M}_{0,A}(X,\beta) \to \overline{M}_{0,A}$ that forget the map μ. In both these cases, as before, one must collapse components that become unstable.

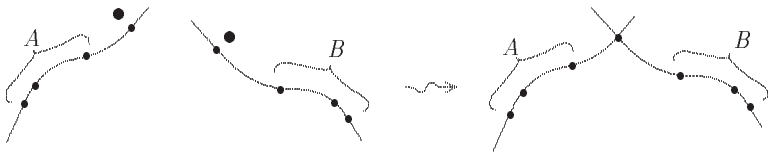
When X is convex, the spaces $\overline{M}_{0,n}(X,\beta)$ have fundamental boundary divisors analogous to the divisors $D(A|B)$ on $\overline{M}_{0,n}$. Let $n \geq 4$. Let $A \cup B$ be a partition of $[n]$. Let $\beta_1 + \beta_2 = \beta$ be a sum in A_1X. There is a divisor in $\overline{M}_{0,n}(X,\beta)$ determined by:

$$D(A,B;\beta_1,\beta_2) = \overline{M}_{0,A\cup\{\bullet\}}(X,\beta_1) \times_X \overline{M}_{0,B\cup\{\bullet\}}(X,\beta_2), \tag{2}$$

$$D(A,B;\beta_1,\beta_2) \subset \overline{M}_{0,n}(X,\beta).$$

A moduli point in $D(A,B,\beta_1,\beta_2)$ corresponds to a map with a reducible domain $C = C_1 \cup C_2$ where $\mu_*([C_1]) = \beta_1$ and $\mu_*([C_2]) = \beta_2$. The points labeled by A lie

on C_1 and points labeled by B lie on C_2. The curves C_1 and C_2 are connected at the points labeled $\bullet$.



Finally, the fiber product in the definition (2) corresponds to the condition that the maps must take the same value in X on the marked point $\bullet$ in order to be glued. This fiber product is defined via evaluation maps discussed in the next section.

For i, j, k, l distinct in $[n]$, set

$$D(i, j \mid k, l) = \sum D(A, B; \beta_1, \beta_2).$$

The sum is over all partitions $A \cup B = [n]$ with $i, j \in A$ and $k, l \in B$ and over all classes $\beta_1, \beta_2 \in A_1 X$ satisfying $\beta_1 + \beta_2 = \beta$. Using the projection $\overline{M}_{0,n}(X, \beta) \to \overline{M}_{0,\{i,j,k,l\}} \stackrel{\sim}{=} \mathbf{P}^1$, the fundamental linear equivalences

$$D(i, j \mid k, l) = D(i, k \mid j, l) = D(i, l \mid j, k) \tag{3}$$

on $\overline{M}_{0,n}(X, \beta)$ are obtained via pull-back of the the 4-point linear equivalences on $\overline{M}_{0,\{i,j,k,l\}}$ as in (1).

0.5. Gromov-Witten invariants and quantum cohomology. Let X be a convex variety. For each marked point $1 \leq i \leq n$, there is a canonical *evaluation map*

$$\rho_i : \overline{M}_{0,n}(X, \beta) \to X$$

defined for $[C, p_1, \ldots, p_n, \mu]$ in $\overline{M}_{0,n}(X, \beta)$ by:

$$\rho_i([C, p_1, \ldots, p_n, \mu]) = \mu(p_i).$$

Given classes $\gamma_1, \ldots, \gamma_n$ in $A^* X$, a product is determined in the ring $A^*(\overline{M}_{0,n}(X, \beta))$ by:

$$\rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n) \in A^*(\overline{M}_{0,n}(X, \beta)). \tag{4}$$

If $\sum \operatorname{codim}(\gamma_i) = \dim(\overline{M}_{0,n}(X, \beta))$, the product (4) can be evaluated on the fundamental class of $\overline{M}_{0,n}(X, \beta)$. In this case, the *Gromov-Witten invariant* $I_\beta(\gamma_1 \cdots \gamma_n)$ is defined by:

$$I_\beta(\gamma_1 \cdots \gamma_n) = \int_{\overline{M}_{0,n}(X,\beta)} \rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n). \tag{5}$$

The multiplicative notation in the argument of I_β is used to indicate I_β is a symmetric function of the classes $\gamma_1, \ldots, \gamma_n$.

Let X be a homogeneous space. Poincaré duality and Bertini-type transversality arguments imply a relationship between the Gromov-Witten invariants and enumerative geometry. If $\gamma_i = [V_i]$ for a subvariety $V_i \subset X$, the Gromov-Witten invariant (5) equals the number of marked rational curves in X with i^{th} marked point in V_i, suitably counted. For example, when $X = \mathbf{P}^2$, $\beta = d[\text{line}]$, $n = 3d - 1$, and each V_i is a point,

$$N_d = I_d(\underbrace{[p] \cdots [p]}_{3d-1}).$$

The Gromov-Witten invariants are used to define the quantum cohomology ring. Associativity of this ring is established as a consequence of the 4-point linear equivalences (3). Associativity amounts to many equations among the Gromov-Witten invariants which often lead to a determination of all the invariants in terms of a few basic numbers. In these notes, the quantum cohomology ring is constructed for homogeneous spaces X. This assumption on X allows for many technical simplifications.

Given $\gamma_1, \ldots, \gamma_n \in H^*X$ (not necessarily of even degrees), there are more general Gromov-Witten invariants in $H^*\overline{M}_{0,n}$ defined by

$$I^X_{0,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n) = \eta_*(\rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n))$$

where $\eta : \overline{M}_{0,n}(X,\beta) \to \overline{M}_{0,n}$ is the projection. The set of multilinear maps

$$\{I^X_{0,n,\beta} : (H^*X)^{\otimes n} \to H^*\overline{M}_{0,n}\}$$

is called the *Tree-Level System of Gromov-Witten Invariants*. We will not need these generalities here.

The construction and proofs of the basic properties of $\overline{M}_{0,n}(X,\beta)$ are undertaken in sections 1–6. The theory of Gromov-Witten invariants and quantum cohomology is presented in sections 7–10 with the examples of $\mathbf{P}^2$, $\mathbf{P}^3$, and a smooth quadric 3-fold $\mathbf{Q}^3$ worked out in full detail. If Theorems 1–3 are taken for granted, sections 1–6 can be skipped. No originality is claimed for these notes except for some aspects of the proofs of Theorems 1–4. Constructions of Kontsevich's moduli space of stable maps can also be found in [A], [K], [B-M]. In [A], a generalization to the case in which the domain is a surface is analyzed.

0.6. Calculation of N_d. We end this introduction by sketching how these moduli spaces of maps can be used to calculate the number N_d of degree d rational plane curves passing through $3d-1$ general points in $\mathbf{P}^2$. The formula (6) will be recovered in section 9 from the general quantum cohomology results, but it may be useful now to see a direct proof.

For $d = 1$, $N_1 = 1$ is the number of lines through 2 points. N_d is determined for $d \geq 2$ by the *recursion formula*:

$$N_d = \sum_{d_1+d_2=d,\ d_1,d_2>0} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right). \tag{6}$$

For example, (6) yields[1] :

$$N_2 = 1,\ N_3 = 12,\ N_4 = 620,\ N_5 = 87304,\ N_6 = 26312976, \ldots$$

The strategy of proof is to utilize the fundamental linear relations (3) among boundary components of $\overline{M}_{0,n}(\mathbf{P}^2, d)$. Intersection of a curve Y in this moduli space with the linear equivalence (3) will yield (6). We will take $n = 3d$ (not $3d-1$) with $d \geq 2$, so $n \geq 6$. Label the marked points by the set

$$\{1, 2, 3, \ldots, n-4, q, r, s, t\}.$$

The forgetful morphism $\overline{M}_{0,n}(\mathbf{P}^2, d) \to \overline{M}_{0,\{q,r,s,t\}}$ yields the relations (3) on $\overline{M}_{0,n}(\mathbf{P}^2, d)$:

$$D(q, r \mid s, t) = D(q, s \mid r, t). \tag{7}$$

[1] Z. Ran reports $N_4 = 620$ as well as the higher N_d 's can be derived from his formulas in [R1]; see [R2] for a comparison of the two approaches.

Recall from section 0.4:

$$D(q, r \mid s, t) = \sum_{q,r \in A,\ s,t \in B,\ d_1+d_2=d} D(A, B; d_1, d_2).$$

The curve $Y \subset \overline{M}_{0,n}(\mathbf{P}^2, d)$ is determined by a selection of general points and lines in $\mathbf{P}^2$. More precisely, let $z_1, \ldots, z_{n-4}, z_s, z_t$ be $n-2$ general points in $\mathbf{P}^2$ and let l_q, l_r be general lines. Let the curve Y be defined by the intersection:

$$Y = \rho_1^{-1}(z_1) \cap \cdots \cap \rho_{n-4}^{-1}(z_{n-4}) \cap \rho_q^{-1}(l_q) \cap \rho_r^{-1}(l_r) \cap \rho_s^{-1}(z_s) \cap \rho_t^{-1}(z_t).$$

$\overline{M}_{0,n}(\mathbf{P}^2, d)$ is a nonsingular, fine moduli space on the open set of automorphism-free maps (see section 1.2). It is not difficult to show the locus of maps with non-trivial automorphisms in $\overline{M}_{0,n}(\mathbf{P}^2, d)$ is of codimension at least 2 if $(n, d) \neq (0, 2)$. Therefore, by Bertini's Theorem applied to each evaluation map and the generality of the points and lines, we conclude Y is a nonsingular curve contained in the automorphism-free locus which intersects all the boundary divisors transversally at general points of the boundary. It remains only to compute the intersection of Y with each side of the linear equivalence (7).

The points of

$$Y \cap\ D(A, B; d_1, d_2)$$

correspond bijectively to maps $\mu : C = C_A \cup C_B \to \mathbf{P}^2$ satisfying:

(a) $C_A, C_B \stackrel{\sim}{=} \mathbf{P}^1$ and meet transversally at a point.
(b) The markings of A, B lie on C_A, C_B respectively.
(c) $\mu_*([C_A]) = d_1[\text{line}]$, $\mu_*([C_B]) = d_2[\text{line}]$.
(d) $\forall 1 \leq i \leq n-4,\ \mu(i) = z_i$.
(e) $\mu(q) \in l_q,\ \mu(r) \in l_r,\ \mu(s) = z_s,\ \mu(t) = z_t$.

Let $q, r \in A$ and $s, t \in B$. $Y \cap\ D(A, B; 0, d)$ is nonempty only when $A = \{q, r\}$. In this case, C_A is required to map to the point $l_q \cap l_r$. The restriction $\mu : C_B \to \mathbf{P}^2$ must map the $3d-2$ markings on C_B to the $3d-2$ given points, and in addition, μ maps the point $C_A \cap C_B$ to $l_q \cap l_r$. Therefore,

$$\#\ Y \cap\ D(\{q, r\}, \{1, \ldots, n-4, s, t\}; 0, d) = N_d.$$

For $1 \leq d_1 \leq d-1$, $Y \cap D(A, B; d_1, d_2)$ is nonempty only when $|A| = 3d_1 + 1$. There are $\binom{3d-4}{3d_1-1}$ partitions satisfying $q, r \in A$, $s, t \in B$, and $|A| = 3d_1 + 1$. A simple count of maps satisfying (a)-(e) yields

$$\#\ Y \cap\ D(A, B; d_1, d_2) = N_{d_1} N_{d_2} d_1^3 d_2$$

for each partition. There are N_{d_1} choices for the image of C_A and N_{d_2} choices for the image of C_B. The points labeled q and r map to any of the d_1 intersection points of $\mu(C_A)$ with l_q and l_r respectively. Finally, there are $d_1 d_2$ choices for the image of the intersection point $C_A \cap C_B$ corresponding to the intersection points of $\mu(C_A) \cap \mu(C_B) \subset \mathbf{P}^2$. The last case is simple: $Y \cap\ D(A, B; d, 0) = \emptyset$. Therefore,

$$\#\ Y \cap\ D(q, r \mid s, t) = N_d + \sum_{d_1+d_2=d,\ d_1>0,\ d_2>0} N_{d_1} N_{d_2} d_1^3 d_2 \binom{3d-4}{3d_1-1}.$$

Now consider the other side of the linear equivalence (7). Let the markings now satisfy $q, s \in A$ and $r, t \in B$. $Y \cap\ D(A, B; 0, d)$ and $Y \cap\ (A, B; d, 0)$ are both

empty. For $1 \leq d_1 \leq d-1$, $Y \cap\ (A \cup B, d_1, d_2)$ is nonempty only when $|A| = 3d_1$. There are $\binom{3d-4}{3d_1-2}$ such partitions and

$$\#\ Y \cap\ D(A, B; d_1, d_2) = N_{d_1} N_{d_2} d_1^2 d_2^2$$

for each. Therefore,

$$\#\ Y \cap\ D(q, s \mid r, t) = \sum_{d_1+d_2=d,\ d_1>0,\ d_2>0} N_{d_1} N_{d_2} d_1^2 d_2^2 \binom{3d-4}{3d_1-2}.$$

The linear equivalence (7) implies

$$\#\ Y \cap\ D(q, r \mid s, t) = \#\ Y \cap\ D(q, s \mid r, t).$$

The recursion (6) follows immediately.

In the general development of quantum cohomology described in sections 8 and 9, these numerical relations obtained by intersection with the basic linear equivalences arise as ring associativity relations.

1. Stable maps and their moduli spaces

1.1. Definitions. An n-pointed, genus g, complex, *quasi-stable* curve

$$(C,\ p_1, \ldots, p_n)$$

is a projective, connected, reduced, (at worst) nodal curve of arithmetic genus g with n distinct, nonsingular, marked points. Let S be an algebraic scheme over $\mathbb{C}$. A *family* of n-pointed, genus g, quasi-stable curves over S is a flat, projective map $\pi : \mathcal{C} \to S$ with n sections $p_1, \ldots, p_n$ such that each geometric fiber $(\mathcal{C}_s,\ p_1(s), \ldots, p_n(s))$ is an n-pointed, genus g, quasi-stable curve. Let X be an algebraic scheme over $\mathbb{C}$. A *family of maps* over S from n-pointed, genus g curves to X consists of the data $(\pi : \mathcal{C} \to S, \{p_i\}_{1 \leq i \leq n}\ , \mu : \mathcal{C} \to X)$:

(i) A family of n-pointed, genus g, quasi-stable curves $\pi : \mathcal{C} \to S$ with n sections $\{p_1, \ldots, p_n\}$.
(ii) A morphism $\mu : \mathcal{C} \to X$.

Two families of maps over S,

$$(\pi : \mathcal{C} \to S, \{p_i\}, \mu),\quad (\pi' : \mathcal{C}' \to S, \{p_i'\}, \mu'),$$

are *isomorphic* if there exists a scheme isomorphism $\tau : \mathcal{C} \to \mathcal{C}'$ satisfying: $\pi = \pi' \circ \tau$, $p_i' = \tau \circ p_i$, $\mu = \mu' \circ \tau$. When $\pi : C \to \mathrm{Spec}(\mathbb{C})$ is the structure map, $(\pi : C \to \mathrm{Spec}(\mathbb{C}), \{p_i\}, \mu)$ is written as $(C, \{p_i\}, \mu)$.

Let $(C, \{p_i\}, \mu)$ be a map from an n-pointed quasi-stable curve to X. The *special points* of an irreducible component $E \subset C$ are the marked points and the component intersections of C that lie on E. The map $(C, \{p_i\}, \mu)$ is *stable* if the following conditions hold for every component $E \subset C$:

(1) If $E \cong \mathbf{P}^1$ and E is mapped to a point by μ, then E must contain at least three special points.
(2) If E has arithmetic genus 1 and E is mapped to a point by μ, then E must contain at least one special point.

A family of pointed maps $(\pi : \mathcal{C} \to S, \{p_i\}, \mu)$ is *stable* if the pointed map on each geometric fiber of π is stable.

If $X = \mathbf{P}^r$, stability can be expressed in the following manner. Let $\omega_{\mathcal{C}/S}$ denote the relative dualizing sheaf. A flat family of maps $(\pi : \mathcal{C} \to S,\ \{p_i\}, \mu)$ is *stable* if and only if $\omega_{\mathcal{C}/S}(p_1 + \ldots + p_n) \otimes \mu^*(\mathcal{O}_{\mathbf{P}^r}(3))$ is π-relatively ample.

Let X be a algebraic scheme over $\mathbb{C}$. Let $\beta \in A_1 X$. A map $\mu : C \to X$ *represents* β if the μ-push-forward of the fundamental class $[C]$ equals β. Define a contravariant functor $\overline{\mathcal{M}}_{g,n}(X,\beta)$ from the category of complex algebraic schemes to sets as follows. Let $\overline{\mathcal{M}}_{g,n}(X,\beta)(S)$ be the set of isomorphism classes of stable families over S of maps from n-pointed, genus g curves to X representing the class β.

1.2. Existence. Let X be a projective, algebraic scheme over $\mathbb{C}$. Projective coarse moduli spaces of maps exist for general g. In the genus 0 case, if X is a projective, nonsingular, convex variety, the coarse moduli spaces are locally normal varieties with finite quotient singularities.

THEOREM 1. *There exists a projective, coarse moduli space* $\overline{M}_{g,n}(X,\beta)$.

$\overline{M}_{g,n}(X,\beta)$ is a scheme together with a natural transformation of functors

$$\phi : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathcal{H}om_{Sch}(*, \overline{M}_{g,n}(X,\beta))$$

satisfying properties:

(I) $\phi(\mathrm{Spec}(\mathbb{C})) : \overline{\mathcal{M}}_{g,n}(X,\beta)(\mathrm{Spec}(\mathbb{C})) \to \mathcal{H}om(\mathrm{Spec}(\mathbb{C}), \overline{M}_{g,n}(X,\beta))$ is a set bijection.

(II) If Z is a scheme and $\psi : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathcal{H}om(*,Z)$ is a natural transformation of functors, then there exists a unique morphism of schemes

$$\gamma : \overline{M}_{g,n}(X,\beta) \to Z$$

such that $\psi = \tilde{\gamma} \circ \phi$. ($\tilde{\gamma} : \mathcal{H}om(*, \overline{M}_{g,n}(X,\beta)) \to \mathcal{H}om(*,Z)$ is the natural transformation induced by γ.)

Let $(C, \{p_i\}, \mu)$ be a map of an n-pointed, quasi-stable curve to X. An automorphism of the map is an automorphism, τ, of the curve C satisfying

$$p_i = \tau(p_i), \quad \mu = \mu \circ \tau.$$

It is straightforward to check that $(C, \{p_i\}, \mu)$ is stable if and only if $(C, \{p_i\}, \mu)$ has a finite automorphism group. Let $\overline{M}^*_{g,n}(X,\beta) \subset \overline{M}_{g,n}(X,\beta)$ denote the open locus of stable maps with no non-trivial automorphisms.

A nonsingular variety X is *convex* if for every map $\mu : \mathbf{P}^1 \to X$,

$$H^1(\mathbf{P}^1, \mu^*(T_X)) = 0$$

where T_X is the tangent bundle to X. If the T_X is generated by global sections, X is convex. Hence, any variety admitting a transitive group action is convex. The second and third theorems concern the convex, genus 0 case.

THEOREM 2. *Let X be a projective, nonsingular, convex variety.*

(i) $\overline{M}_{0,n}(X,\beta)$ *is a locally normal, projective variety of pure dimension*

$$\dim(X) + \int_\beta c_1(T_X) + n - 3.$$

(ii) $\overline{M}_{0,n}(X,\beta)$ *is locally a quotient of a nonsingular variety by a finite group.*

(iii) $\overline{M}^*_{0,n}(X,\beta)$ *is a nonsingular, fine moduli space (for automorphism-free stable maps) equipped with a universal family.*

In fact, if the language of stacks is pursued, it can be seen that the moduli problem of stable maps from n-pointed, genus 0 curves to a nonsingular, convex space X determines a complete, nonsingular, algebraic stack. For simplicity, the stack theoretic view is not taken in these notes; the experienced reader will see how to make the required modifications.

The *boundary* of $\overline{M}_{0,n}(X,\beta)$ is the locus corresponding to reducible domain curves. The boundary of the fine moduli space $\overline{M}_{0,n}$ is a divisor with normal crossings. In the coarse moduli spaces $\overline{M}_g$ and $\overline{M}_{g,n}$, the boundary is a divisor with normal crossings modulo a finite group. $\overline{M}_{0,n}(X,\beta)$ has the same boundary singularity type as these moduli spaces of pointed curves.

THEOREM 3. *Let X be a nonsingular, projective, convex variety. The boundary of $\overline{M}_{0,n}(X,\beta)$ is a divisor with normal crossings (up to a finite group quotient).*

The organization of the construction is as follows. First $\overline{M}_{g,n}(\mathbf{P}^r,d)$ is explicitly constructed in sections 2–4. If $X \subset \mathbf{P}^r$ is a closed subscheme, it is not difficult to define a natural, closed subscheme

$$\overline{M}_{g,n}(X,d) \subset \overline{M}_{g,n}(\mathbf{P}^r,d)$$

of maps that factor through X. $\overline{M}_{g,n}(X,d)$ is a disjoint union of the spaces $\overline{M}_{g,n}(X,\beta)$ as β varies in A_1X. By the universal property, it can be seen that the coarse moduli spaces $\overline{M}_{g,n}(X,\beta)$ do not depend on the projective embedding of X (see section 5). The deformation arguments required to deduce Theorem 3 from the convexity assumption are covered in section 5. The boundary of the space of maps is discussed in section 6.

1.3. Natural structures. The universal property of the moduli space of maps immediately yields geometric structures on $\overline{M}_{g,n}(X,\beta)$. Consider first the marked points. The n marked points induce n canonical evaluation maps $\rho_1,\ldots,\rho_n$ on $\overline{M}_{g,n}(X,\beta)$. For $1 \leq i \leq n$, define a natural transformation

$$\theta_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathcal{H}om(*,X)$$

as follows. Let $\zeta = (\pi : \mathcal{C} \to S,\ \{p_i\},\ \mu)$ be an element of $\overline{\mathcal{M}}_{g,n}(X,\beta)(S)$. Let

$$\theta_i(S)(\zeta) = \mu \circ p_i \in \mathcal{H}om(S,X).$$

θ_i is easily seen to be a natural transformation. By Theorem 1, θ_i induces a unique morphism of schemes $\rho_i : \overline{M}_{g,n}(X,\beta) \to X$.

By the universal properties of the moduli spaces $\overline{M}_{g,n}$ of n-pointed Deligne-Mumford stable genus g curves (in case $2g-2+n > 0$), each element $\zeta \in \overline{\mathcal{M}}_{g,n}(X,\beta)(S)$ naturally yields a morphism $S \to \overline{M}_{g,n}$ ([Kn]). Therefore, there exist natural forgetful maps $\eta : \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}$.

2. Boundedness and a quotient approach

2.1. Summary. In this section, the case $X = \mathbf{P}^r$ will be considered. The boundedness of the moduli problem of pointed stable maps is established. The arguments lead naturally to a quotient approach to the coarse moduli space. To set up the quotient approach, a result on equality loci of families of line bundles is required.

2.2. Equality of line bundles in families. Results on scheme theoretic equality loci are recalled. Let $\pi : \mathcal{C} \to S$ be a flat family of quasi-stable curves. By the theorems of cohomology and base change (cf. [H]), there is a canonical isomorphism $\mathcal{O}_S \cong \pi_*(\mathcal{O}_\mathcal{C})$. Hence, for any line bundle $\mathcal{N}$ on S, there is a canonical isomorphism $\mathcal{N} \cong \pi_*\pi^*(\mathcal{N})$. Suppose $\mathcal{L}$ and $\mathcal{M}$ are two line bundles on $\mathcal{C}$. The existence of a line bundle $\mathcal{N}$ on S such that $\mathcal{L} \otimes \mathcal{M}^{-1} \cong \pi^*(\mathcal{N})$ is equivalent to the joint validity of (a) and (b):

(a) $\pi_*(\mathcal{L} \otimes \mathcal{M}^{-1})$ is locally free.
(b) The canonical map $\pi^*\pi_*(\mathcal{L} \otimes \mathcal{M}^{-1}) \to \mathcal{L} \otimes \mathcal{M}^{-1}$ is an isomorphism.

Let $\mathcal{L}_s$ be a line bundle on the the geometric fiber $\mathcal{C}_s$ of π. The *multidegree* of $\mathcal{L}_s$ assigns to each irreducible component of $\mathcal{C}_s$ the degree of the restriction of $\mathcal{L}_s$ to that component.

PROPOSITION 1. *Let $\mathcal{L}$, $\mathcal{M}$ be line bundles on $\mathcal{C}$ such that the multidegrees of $\mathcal{L}_s$ and $\mathcal{M}_s$ coincide on each geometric fiber $\mathcal{C}_s$. Then, there is a unique closed subscheme $T \to S$ satisfying the following two properties:*

(I) *There is a line bundle $\mathcal{N}$ on T such that $\mathcal{L}_T \otimes \mathcal{M}_T^{-1} \cong \pi^*(\mathcal{N})$.*
(II) *If $(R \to S,\ \mathcal{N})$ is a pair of a morphism from R to S and a line bundle on R such that $\mathcal{L}_R \otimes \mathcal{M}_R^{-1} \cong \pi^*(\mathcal{N})$, then $R \to S$ factors through T.*

Proof. The proof of the Theorem of the Cube (II) in [M1] also establishes this proposition. The multidegree condition implies $\mathcal{L}_s \cong \mathcal{M}_s$ if and only if $h^0(\mathcal{C}_s, \mathcal{L}_s \otimes \mathcal{M}_s^{-1}) = 1$. The multidegree condition is required for T to be a *closed* subscheme. □

2.3. Boundedness. Let $(C, \{p_i\}, \mu)$ be a stable map from an n-pointed, genus g curve to $\mathbf{P}^r$. Let

$$\mathcal{L} = \omega_C(p_1 + \ldots + p_n) \otimes \mu^*(\mathcal{O}_{\mathbf{P}^r}(3)).$$

$\mathcal{L}$ is ample on C. A simple argument shows there exists an $f = f(g, n, r, d) > 0$ such that $\mathcal{L}^f$ is very ample on C and $h^1(C, \mathcal{L}^f) = 0$, so

$$\text{degree}(\mathcal{L}^f) = f \cdot (2g - 2 + n + 3d) = e\,,$$
$$h^0(C, \mathcal{L}^f) = e - g + 1.$$

Let $W \cong \mathbb{C}^{e-g+1}$ be a vector space. An isomorphism

$$W^* \xrightarrow{\sim} H^0(C, \mathcal{L}^f) \tag{8}$$

induces embeddings $\iota : C \hookrightarrow \mathbf{P}(W)$ and $\gamma : C \hookrightarrow \mathbf{P}(W) \times \mathbf{P}^r$ where $\gamma = (\iota, \mu)$. The n sections $\{p_i\}$ yield n points $(\iota \circ p_i, \mu \circ p_i)$ of $\mathbf{P}(W) \times \mathbf{P}^r$. Let H be the Hilbert scheme of genus g curves in $\mathbf{P}(W) \times \mathbf{P}^r$ of multidegree (e, d). Let $P_i = \mathbf{P}(W) \times \mathbf{P}^r$ be the Hilbert scheme of a point in $\mathbf{P}(W) \times \mathbf{P}^r$. Via the isomorphism (8), a point in $H \times P_1 \times \ldots \times P_n$ is associated to the stable map $(C, \{p_i\}, \mu)$.

The locus of points in $H \times P_1 \times \ldots \times P_n$ corresponding to stable maps has a natural quasi-projective scheme structure. There is a natural closed incidence subscheme

$$I \subset H \times P_1 \times P_2 \times \ldots \times P_n$$

corresponding to the locus where the n points lie on the curve. There is an open set $U \subset I$ satisfying the following:

(i) The curve C is quasi-stable.

(ii) The natural projection $C \to \mathbf{P}(W)$ is a non-degenerate embedding.
(iii) The n points lie in the nonsingular locus of C.
(iv) The multidegree of $\mathcal{O}_{\mathbf{P}(W)}(1) \otimes \mathcal{O}_{\mathbf{P}^r}(1)|_C$ equals the multidegree of

$$\omega_C^f(fp_1 + fp_2 + \ldots + fp_n) \otimes \mathcal{O}_{\mathbf{P}^r}(3f+1)|_C.$$

By Proposition 1, there exists a natural closed subscheme $J \subset U$ where the line bundles of condition (iv) above coincide. J corresponds to the locus of stable maps. The natural $PGL(W)$-action on $\mathbf{P}(W) \times \mathbf{P}^r$ yields $PGL(W)$-actions on H, P_i, I, U, and J. To each stable map from an n-pointed, genus g curve to $\mathbf{P}^r$, we have associated a $PGL(W)$-orbit in J. If two stable maps are associated to the same orbit, the two stable maps are isomorphic. The stability condition implies that a stable map has no infinitesimal automorphisms. It follows that the $PGL(W)$-action on J has finite stabilizers.

2.4. Quotients. The moduli space of stable maps is $J/PGL(W)$. It may be possible to construct the quotient $J/PGL(W)$ via Geometric Invariant Theory. Another method will be pursued here. The quotient will be first constructed as a proper, algebraic variety by using auxiliary moduli spaces of pointed curves. Projectivity will then be established via J. Kollár's semipositivity approach.

3. A rigidification of $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d)$

3.1. Review of Cartier divisors. An effective Cartier divisor D on a scheme Y is a closed subscheme that is locally defined by a non-zero-divisor. An effective Cartier divisor determines a line bundle $\mathcal{L} = \mathcal{O}(D)$ together with a section $s \in H^0(Y, \mathcal{L})$ locally not a zero-divisor such that D is the subscheme defined by $s = 0$. (As an invertible sheaf, $\mathcal{O}(D)$ can be constructed as the subsheaf of rational functions with at most simple poles along D with s equal to the function 1, see [M2].) Conversely, if the pair $(\mathcal{L}, s)$ satisfies:

(i) $\mathcal{L}$ is line bundle on Y.
(ii) $s \in H^0(Y, \mathcal{L})$ is a section locally not a zero divisor.

then the zero scheme of s is an effective Cartier divisor on Y.

Lemma 1. *Let the pairs $(\mathcal{L}, s)$ and $(\mathcal{L}', s')$ satisfy (i) and (ii) above. If the two pairs yield the same Cartier divisor, then there exists a unique isomorphism $\mathcal{L} \to \mathcal{L}'$ taking s to s'.*

3.2. Definitions. We assume throughout the construction that $r > 0$, $d > 0$, and $(g, n, r, d) \neq (0, 0, 1, 1)$. If $r = 0$, the functor of stable maps to $\mathbf{P}^0$ is coarsely represented by $\overline{M}_{g,n}$. If $d = 0$, the functor $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, 0)$ is coarsely represented by $\overline{M}_{g,n} \times \mathbf{P}^r$ and, $\overline{M}_{0,0}(1,1)$ is easily seen to be $\mathrm{Spec}(\mathbb{C})$. For all other values, the construction of $\overline{M}_{g,n}(X, \beta)$ will be undertaken.

Let $V^* = H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(1))$. Then $\mathbf{P}^r = \mathbf{P}(V)$. Let $\overline{t} = (t_0, \ldots, t_r)$ span a basis of V^*. A *$\overline{t}$-rigid stable* family of degree d maps from n-pointed, genus g curves to $\mathbf{P}^r$ consists of the data

$$(\pi : \mathcal{C} \to S,\ \{p_i\}_{1 \leq i \leq n}\ ,\ \{q_{i,j}\}_{0 \leq i \leq r,\ 1 \leq j \leq d}\ ,\ \mu)$$

where:

(i) $(\pi : \mathcal{C} \to S,\ \{p_i\}, \mu)$ is a stable family of degree d maps from n-pointed, genus g curves to $\mathbf{P}^r$.

(ii) $(\pi : \mathcal{C} \to S,\ \{p_i\}, \{q_{i,j}\})$ is a flat, projective family of $n + d(r+1)$-pointed, genus g, Deligne-Mumford stable curves with sections $\{p_i\}$ and $\{q_{i,j}\}$.

(iii) For $0 \leq i \leq r$, there is an equality of Cartier divisors

$$\mu^*(t_i) = q_{i,1} + q_{i,2} + \ldots + q_{i,d}.$$

Condition (iii) implies each fibered map of the family intersects each hyperplane $(t_i) \subset \mathbf{P}^r$ transversally. Condition (ii) guarantees these hyperplane intersections are unmarked, nonsingular points.

If $(g, n, r, d) = (0, 0, 1, 1)$, then $n + d(r+1) = 2$. There are no Deligne-Mumford stable 2-pointed genus 0 curves. This is why $(0, 0, 1, 1)$ is avoided.

Define a contravariant functor $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d, \overline{t})$ from the category of complex algebraic schemes to sets as follows. Let $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d, \overline{t})(S)$ be the set of isomorphism classes of $\overline{t}$-rigid stable families over S of degree d maps from n-pointed, genus g curves to $\mathbf{P}^r$. Note that the functor $\overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d, \overline{t})$ depends only upon the spanning hyperplanes $(t_i) \subset \mathbf{P}^r$ and not upon the additional $\mathbb{C}^*$-choices in the defining equations t_i of the hyperplanes. Nevertheless, it is natural for the following constructions to consider the equations of the hyperplanes $\overline{t} = (t_0, \ldots, t_r)$.

Proposition 2. *There exists a quasi-projective coarse moduli space,*

$$\overline{M}_{g,n}(\mathbf{P}^r, d, \overline{t}),$$

and a natural transformation of functors

$$\psi : \overline{\mathcal{M}}_{g,n}(\mathbf{P}^r, d, \overline{t}) \to \mathcal{H}om(*, \overline{M}_{g,n}(\mathbf{P}^r, d, \overline{t}))$$

satisfying the analogous conditions (I) and (II) of Theorem 1.

The genus 0 case is simpler.

Proposition 3. $\overline{M}_{0,n}(\mathbf{P}^r, d, \overline{t})$ *represents the functor* $\overline{\mathcal{M}}_{0,n}(\mathbf{P}^r, d, \overline{t})$ *and is a nonsingular algebraic variety.*

3.3. Proofs. A complete proof of Proposition 3 will be given. The proof of Proposition 2 is almost identical. Remarks indicating the differences will be made. The dependence of the coarse and fine moduli property on the genus in Propositions 2 and 3 is a direct consequence of the fact that $\overline{M}_{g,n}$ is a coarse moduli space for $g > 0$ and a fine moduli space for $g = 0$.

The idea behind the construction is the following. Let $m = n + d(r+1)$. The data of the $\overline{t}$-rigid stable family immediately yields a morphism of the base S to $\overline{M}_{g,m}$. In fact, the image of S lies in a universal, locally closed subscheme of $\overline{M}_{g,m}$. This subscheme is denoted by B. The first step of the construction is to identify B. The morphism $S \to B$ does not contain all the data of the $\overline{t}$-rigid stable family. Consider the case in which the base S is a point. The corresponding point in B records the domain curve C, the marked points $\{p_i\}$, and the pull-back divisors under μ of the hyperplanes in $\mathbf{P}^r$ determined by $\overline{t}$. The map μ is determined by the pull-back divisors up to the diagonal torus action on $\mathbf{P}^r$. The torus information is recorded in the total space of r tautological $\mathbb{C}^*$-bundles over B. The $\overline{t}$-stable moduli space is expressed as the total space of these r distinct $\mathbb{C}^*$-bundles over B. To canonically construct the universal family over the $\overline{t}$-stable moduli space, the equations t_i of the hyperplanes are needed. This is why the equations t_i (rather than the spanning hyperplanes (t_i)) are explicitly chosen.

Proposition 3 is proved by an explicit construction of $\overline{M}_{0,n}(\mathbf{P}^r, d, \overline{t})$ together with a universal family of $\overline{t}$-rigid stable maps. Let $m = n + d(r+1)$. Let $\overline{M}_{0,m}$ be

the Mumford-Knudsen compactification of the moduli space of m-pointed, genus 0 curves. Let $\pi : \overline{U}_{0,m} \to \overline{M}_{0,m}$ be the universal curve with m sections $\{p_i\}_{1\leq i\leq n}$ and $\{q_{i,j}\}_{0\leq i\leq r,\ 1\leq j\leq d}$. Since $\overline{U}_{0,m}$ is nonsingular and the sections are of codimension 1, there are canonically defined line bundles:

$$\mathcal{H}_i = \mathcal{O}_{\overline{U}_{0,m}}(q_{i,1} + q_{i,2} + \ldots + q_{i,d}),$$

for $0 \leq i \leq r$. Let $s_i \in H^0(\overline{U}_{0,m}, \mathcal{H}_i)$ be the canonical section representing the Cartier divisor $(q_{i,1} + q_{i,2} + \ldots + q_{i,d})$.

For any morphism $\gamma : X \to \overline{M}_{0,m}$, consider the fiber product:

$$\begin{array}{ccc} X \times_{\overline{M}_{0,m}} \overline{U}_{0,m} & \xrightarrow{\overline{\gamma}} & \overline{U}_{0,m} \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\gamma} & \overline{M}_{0,m} \end{array}$$

We call the morphism $\gamma : X \to \overline{M}_{0,m}$ $\mathcal{H}$-*balanced* if

(a) For $1 \leq i \leq r$, $\pi_{X*}\overline{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ is locally free.

(b) For $1 \leq i \leq r$, the canonical map

$$\pi_X^*\pi_{X*}\overline{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}) \to \overline{\gamma}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$$

is an isomorphism.

If γ is $\mathcal{H}$-balanced, the line bundles $\overline{\gamma}^*(\mathcal{H}_i)$ are isomorphic on the fibers of π_X. Let $B \subset \overline{M}_{0,m}$ be the universal, locally closed subscheme satisfying the two following properties:

(i) The inclusion $\iota : B \hookrightarrow \overline{M}_{0,m}$ is $\mathcal{H}$-balanced.

(ii) Every $\mathcal{H}$-balanced morphism $\gamma : X \to \overline{M}_{0,m}$ factors (uniquely) through B.

By Proposition 1, B exists. In fact, $B \subset \overline{M}_{0,m}$ is a Zariski open subscheme. In the $g > 0$ case, the above constructions exist over the stacks $\overline{M}_{g,m}$ and $\overline{U}_{g,m}$. $B_{g,m}$ is a locally closed substack of $\overline{M}_{g,m}$ of positive codimension.

Let $\mathcal{G}_i = \pi_{B*}\overline{\iota}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$ for $1 \leq i \leq r$. Let $\tau_i : Y_i \to B$ be the total space of the canonical $\mathbb{C}^*$-bundle associated to $\mathcal{G}_i$. Y_i is the affine bundle associated to $\mathcal{G}_i$ minus the zero section. The pull-back $\tau_i^*(\mathcal{G}_i)$ has a tautological non-vanishing section and hence is canonically trivial. Consider the product

$$Y = Y_1 \times_B \times Y_2 \times_B \ldots \times_B Y_r$$

equipped with projections $\rho_i : Y \to Y_i$ and a morphism $\tau : Y \to B$. Form the cartesian square:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\overline{\tau}} & \overline{U}_{0,m} \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\tau} & B \subset \overline{M}_{0,m}. \end{array}$$

The line bundles $\overline{\tau}^*(\mathcal{H}_i)$ for $1 \leq i \leq r$ are canonically isomorphic to $\mathcal{L} = \overline{\tau}^*(\mathcal{H}_0)$ on $\mathcal{U}$ since

$$\overline{\tau}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1}) \cong \pi_Y^*\rho_i^*\tau_i^*(\mathcal{G}_i)$$

and $\tau_i^*(\mathcal{G}_i)$ is canonically trivial.

Via pull-back and the canonical isomorphisms, $\overline{\tau}^*(s_i)$ canonically corresponds to a section of $\mathcal{L}$. Since these $r+1$ sections do not vanish simultaneously, they

define a morphism of $\mu : \mathcal{U} \to \mathbf{P}^r$. The canonical method of obtaining μ is as follows. Define a vector space map $V^* \to H^0(\mathcal{L})$ by sending t_i to $\overline{\tau}^*(s_i)$. The induced surjection $V^* \otimes \mathcal{O} \to \mathcal{L}$ canonically yields a morphism

$$\mu : \mathcal{U} \to \mathbf{P}^r.$$

Note that the equations t_i are used to define the morphism μ. The sections $\{p_i\}$, $\{q_{i,j}\}$ pull back to sections of π_M. We claim that the family

$$(\pi_Y : \mathcal{U} \to Y, \{p_i\}, \{q_{i,j}\}, \mu) \tag{9}$$

is a universal family of $\overline{t}$-rigid stable maps, so $\overline{M}_{0,n}(\mathbf{P}^r, d, \overline{t}) = Y$.

The stability of the family of maps

$$(\pi_Y : \mathcal{U} \to Y, \{p_i\}, \mu) \tag{10}$$

is straightforward. Each fiber C of π_Y is an m-pointed, genus 0 stable curve with markings $\{p_i\}$ and $\{q_{i,j}\}$. Let $E \subset C$ be an irreducible component. There are two cases. First suppose $\dim(\mu(E)) = 0$. Then, $\deg_E(\mu^*(\mathcal{O}_{\mathbf{P}^r}(3))) = 0$. By the transversality condition (iii), E has no markings from the sections $\{q_{i,j}\}$. Since C is a stable m-pointed curve and no $\{q_{i,j}\}$ markings lie on E, $\deg_E(\omega_C(p_1 + \ldots + p_n)) > 0$. Therefore,

$$\deg_E(\omega_C(p_1 + \ldots + p_n) \otimes \mu^*(\mathcal{O}_{\mathbf{P}^r}(3)) = \deg_E(\omega_C(p_1 + \ldots + p_n)) > 0.$$

Next suppose that $\mu(E)$ is a curve. Then,

$$\begin{aligned} \deg_E(\omega_C(p_1 + \ldots + p_n) \otimes \mu^*(\mathcal{O}_{\mathbf{P}^r}(3)) &\geq \deg_E(\omega_C \otimes \mu^*(\mathcal{O}_{\mathbf{P}^r}(3))) \\ &\geq -2 + 3 \\ &> 0. \end{aligned}$$

Hence (10) is a stable family of maps. By construction, it is a $\overline{t}$-rigid stable family.

Finally, it must be shown (9) is universal. Let

$$(\pi : \mathcal{C} \to S, \{p_i\}, \{q_{i,j}\}, \nu) \tag{11}$$

be a family of $\overline{t}$-rigid stable maps. Since $(\pi : \mathcal{C} \to S, \{p_i\}, \{q_{i,j}\})$ is a flat family of m-pointed, genus 0 stable curves, there is an induced map $\lambda : S \to \overline{M}_{0,m}$ such that the pull-back family $S \times_{\overline{M}_{0,m}} \overline{U}_{0,m}$ is canonically isomorphic to $(\pi : \mathcal{C} \to S, \{p_i\}, \{q_{i,j}\})$.

First we show λ is $\mathcal{H}_i$-balanced. The pair $(\overline{\lambda}^*(\mathcal{H}_i), \overline{\lambda}^*(s_i))$ yields the Cartier divisor $q_{i,1} + \ldots + q_{i,d}$ on $\mathcal{C}$. The map ν is induced by a vector space homomorphism $\psi : V^* \to H^0(\mathcal{C}, \nu^*(\mathcal{O}_{\mathbf{P}(V)}(1)))$. Let $z_i = \psi(t_i)$. By condition (iii) of $\overline{t}$-rigid stability, the pair $(\nu^*(\mathcal{O}_{\mathbf{P}(V)}(1)), z_i)$ yields the Cartier divisor $q_{i,1} + \ldots + q_{i,d}$ on $\mathcal{C}$. By Lemma 1, there are *canonical* isomorphisms

$$\overline{\lambda}^*(\mathcal{H}_i) \tilde{=} \nu^*(\mathcal{O}_{\mathbf{P}(V)}(1)) \tag{12}$$

for all i. Hence λ is $\mathcal{H}_i$-balanced.

By the universal property of B, λ factors through B: $\lambda : S \to B$. There are canonical isomorphisms

$$\pi_*(\overline{\lambda}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})) \tilde{=} \lambda^*(\mathcal{G}_i). \tag{13}$$

The canonical isomorphisms (12) yield canonical sections of $\overline{\lambda}^*(\mathcal{H}_i \otimes \mathcal{H}_0^{-1})$. The canonical isomorphisms (13) then yield nowhere vanishing sections of $\lambda^*(\mathcal{G}_i)$ over S. Hence there is a canonical a map $S \to Y$. It is easily checked the pull-back

of the universal family over Y yields a $\overline{t}$-rigid stable family of maps canonically isomorphic to (11).

4. The construction of $\overline{M}_{g,n}(\mathbf{P}^r, d)$

4.1. Gluing. While a given pointed stable map $\mu : C \to \mathbf{P}^r$ may not be rigid for a given basis $\overline{t}$ of $V^* = H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(1))$, the map will be rigid (by Bertini's Theorem) for some choice of basis. The moduli space $\overline{M}_{g,n}(\mathbf{P}^r, d)$ is obtained by gluing together quotients of $\overline{M}_{g,n}(\mathbf{P}^r, d, \overline{t})$ for different choices of bases $\overline{t}$.

For notational convenience, set $\overline{M}(\overline{t}) = \overline{M}_{g,n}(\mathbf{P}^r, d, \overline{t})$. We write $(\pi : \mathcal{U} \to \overline{M}(\overline{t}), \{p_i\}, \{q_{i,j}\}, \mu)$ for the universal family of $\overline{t}$-rigid stable maps in the genus 0 case. If $g > 0$, more care is required.

Let $\mathfrak{S}_d$ denote the symmetric group on d letters. The group

$$G = G_{d,r} = \mathfrak{S}_d \times \ldots \times \mathfrak{S}_d \quad (r+1 \text{ factors})$$

has a natural action on $\overline{M}(\overline{t})$ obtained by permuting the ordering in each of the $r+1$ sets of sections $\{q_{i,1}, \ldots, q_{i,d}\}$, $0 \leq i \leq r$. For any $\sigma \in G$, the family

$$(\pi : \mathcal{U} \to \overline{M}(\overline{t}), \{p_i\}, \{q_{i,\sigma(j)}\}, \mu) \tag{14}$$

is also a $\overline{t}$-rigid family over $\overline{M}(\overline{t})$. By the universal property, the permuted family (14) induces an automorphism of $\overline{M}(\overline{t})$. Since $\overline{M}(\overline{t})$ is quasi-projective, there is a quasi-projective quotient scheme $\overline{M}(\overline{t})/G$.

Let $\overline{t}$ and $\overline{t}'$ be distinct choices of bases of V^*. Let $\mu : \mathcal{U} \to \mathbf{P}^r$ be the universal family over $\overline{M}(\overline{t})$. Let

$$\overline{M}(\overline{t}, \overline{t}') \subset \overline{M}(\overline{t})$$

denote the open locus over which the divisors $\mu^*(t'_0), \ldots, \mu^*(t'_r)$ are étale and disjoint from the sections $\{p_i\}$. The open set $\overline{M}(\overline{t}, \overline{t}')$ is certainly G-invariant. Let $\overline{M}(\overline{t}, \overline{t}')/G$ denote the quasi-projective quotient.

Proposition 4. *There is a canonical isomorphism*

$$\overline{M}(\overline{t}, \overline{t}')/G \cong \overline{M}(\overline{t}', \overline{t})/G.$$

Proof. The divisors $\mu^*(t'_i)$ define an étale Galois cover $\mathcal{E}$ of $\overline{M}(\overline{t}, \overline{t}')$ with Galois group G over which a $\overline{t}'$-rigid stable family is defined. The fiber of $\mathcal{E}$ over $(C, \{p_i\}, \{q_{i,j}\}, \mu)$ is the set of orderings $\{q'_{i,j}\}$ of the points mapped by μ to the hyperplane $(t'_i = 0)$. Therefore there is a map

$$\mathcal{E} \to \overline{M}(\overline{t}') \tag{15}$$

which is easily seen be G-equivariant for the Galois G-action on $\mathcal{E}$ and the $\{q'_{i,j}\}$-permutation G-action on the $\overline{M}(\overline{t}')$. Moreover (15) factors through $\overline{M}(\overline{t}', \overline{t})$. Hence there exists a map of quotients

$$\overline{M}(\overline{t}, \overline{t}') \cong \mathcal{E}/\text{Galois} \to \overline{M}(\overline{t}', \overline{t})/G. \tag{16}$$

The map (16) is G-invariant for the $\{q_{i,j}\}$-permutation action on $\overline{M}(\overline{t}, \overline{t}')$. Therefore (16) descends to $\overline{M}(\overline{t}, \overline{t}')/G \to \overline{M}(\overline{t}', \overline{t})/G$. The inverse is obtained by interchanging $\overline{t}$ and $\overline{t}'$ in the above construction. In fact, there is a natural action of $G \times G$ on $\mathcal{E}$ and canonical isomorphisms $\overline{M}(\overline{t}, \overline{t}')/G \cong \mathcal{E}/(G \times G) \cong \overline{M}(\overline{t}', \overline{t})/G$. □

In case $g > 0$, the coarse moduli spaces $\overline{M}_{g,n}(\mathbf{P}^r, d, \overline{t})$ do not (in general) have universal families. The permutation action of G can be defined on a Hilbert scheme or a stack and then descended to $\overline{M}_{g,n}(\mathbf{P}^r, d, \overline{t})$. The open sets $\overline{M}(\overline{t}', \overline{t})$ and $\overline{M}(\overline{t}, \overline{t}')$ are well defined for $g > 0$ and still satisfy Proposition 4.

The cocycle conditions on triple intersections are easily established. Hence, the schemes $\overline{M}(\overline{t})/G$ canonically patch together along the open sets $\overline{M}(\overline{t}, \overline{t}')/G$ to form the scheme $\overline{M}_{g,n}(\mathbf{P}^r, d)$. The results on boundedness show $\overline{M}_{g,n}(\mathbf{P}^r, d)$ is covered by a finite number of these open sets $\overline{M}(\overline{t})/G$. Hence, $\overline{M}_{g,n}(\mathbf{P}^r, d)$ is an algebraic scheme of finite type over $\mathbb{C}$. The universal properties of $\overline{M}_{g,n}(\mathbf{P}^r, d)$ are easily obtained from the universal properties of the moduli spaces of $\overline{t}$-rigid stable maps.

4.2. Separation and completeness. Let (X, x) be a nonsingular, pointed curve. Let $\iota : X \setminus \{x\} = U \hookrightarrow X$. Let

$$(\pi : \mathcal{C} \to X,\ \{p_i\},\ \mu) \tag{17}$$

$$(\pi' : \mathcal{C}' \to X,\ \{p_i'\},\ \mu') \tag{18}$$

be two families over X of stable maps to $\mathbf{P}^r = \mathbf{P}(V)$.

Proposition 5. *An isomorphism between the families (17) and (18) over U extends to an isomorphism over X.*

Proof. Choose a basis $\overline{t} = (t_0, \ldots, t_r)$ of V^* that intersects the maps $\mu : \mathcal{C}_x \to \mathbf{P}^r$ and $\mu' : \mathcal{C}'_x \to \mathbf{P}^r$ transversally at unmarked, nonsingular points. Since it suffices to prove the isomorphism extends over a local étale cover of (X, x), it can be assumed that the Cartier divisors $\mu^*(t_i)$ and $\mu'^*(t_i)$ split into sections $\{q_{i,j}\}$ and $\{q'_{i,j}\}$ of π and π'. Then $\mathcal{C}$, $\mathcal{C}'$ are Deligne-Mumford stable $m = n + d(r+1)$ pointed curves. Therefore, by the separation property of the functor of Deligne-Mumford stable pointed curves, there exists an isomorphism (of pointed curves) $\tau : \mathcal{C} \to \mathcal{C}'$ over X. Since $\tau \circ \mu'$ and μ agree on an open set, $\tau \circ \mu' = \mu$. □

Proposition 5 and the valuative criterion show $\overline{M}_{g,n}(X, \beta)$ is a separated algebraic scheme.

Properness is also established by the valuative criterion. To complete 1 dimensional families of stable maps, semi-stable reduction techniques for curves are used (as in [K-K-M] and [Ha]).

Proposition 6. *Let $\mathcal{F} = (\pi : \mathcal{C} \to U,\ \{p_i\},\ \mu)$ be a family of stable maps to $\mathbf{P}^r$. There exists a base change $\gamma : (Y, y) \to (X, x)$ satisfying:*

(i.) *$\gamma_W : Y \setminus \{y\} = W \to U$ is étale.*
(ii.) *The pull-back family $\gamma_W^*(\mathcal{F})$ extends to a stable family over (Y, y).*

Proof. First, after restriction to a Zariski open subset of U, it can be assumed that the fibers $\mathcal{C}_\xi$ all have the same number of irreducible components. There may be non-trivial monodromy around the point $x \in X$ in the set of irreducible components of the fibers $\mathcal{C}_\xi$. After a base change (possibly ramified at x), this monodromy can be made trivial. It can therefore be assumed that $\mathcal{F}$ is a union of stable families $\mathcal{F}_j = (\pi_j : \mathcal{C}_j \to U,\ \{p_i^j\}, \{p_i^c\},\ \mu_j)$ where π_j is family of *irreducible*, nodal, projective curves. The markings $\{p_i^j\}$ are the markings of $\mathcal{C}$ that lie on $\mathcal{C}_j$. The marking $\{p_j^c\}$ correspond to intersections of components in $\mathcal{F}$. It suffices to prove Proposition 6 separately for each stable family $\mathcal{F}_j$.

For technical reasons, it is convenient to consider families of nonsingular curves. After restriction, normalization, and base change of $\mathcal{F}_j$, a family

$$(\tilde{\pi}_j : \tilde{\mathcal{C}}_j \to U, \{p_i^j\}, \{p_i^c\}, \{p_i^n\}, \tilde{\mu}_j) \tag{19}$$

can be obtained where $\tilde{\mathcal{F}}_j$ is a family of stable maps of irreducible, *nonsingular*, projective curves. The additional markings $\{p_i^n\}$ correspond to the nodes. Consider the nodal locus in $\mathcal{F}_j$. This locus consists of curves and isolated points. Via restriction of U to a Zariski open set, it can be assumed the nodal locus (if non-empty) is of pure dimension 1. A normalization now separates the sheets along the nodal locus. A base change then may be required to make the separated points $\{p_i^n\}$ sections. If the normalized family $\tilde{\mathcal{F}}_j$ is completed, $\mathcal{F}_j$ can be completed by identifying the nodal markings on $\tilde{\mathcal{F}}_j$. This nodal identification commutes with the map to $\mathbf{P}^r$. It therefore suffices to prove Proposition 6 for these normalized families (19).

By the above reductions, it suffices to prove Proposition 6 for a family of stable maps of irreducible, nonsingular, projective curves. Let

$$(\pi : \mathcal{C} \to U, \{p_i\}, \mu) \tag{20}$$

be such a family. Let $\pi : \mathcal{E} \to X$ be a flat extension of $\pi : \mathcal{C} \to U$ over the point $x \in X$. After blow-ups in the special fiber of $\mathcal{E}$, it can be assumed the map $\mu : \mathcal{C} \to \mathbf{P}^r$ extends to $\mu : \mathcal{E} \to \mathbf{P}^r$. By Lemma 2 below applied to the flat extension $\pi : \mathcal{E} \to X$, there exists a base change $\gamma : (Y, y) \to (X, x)$ and a family of pointed curves

$$\pi_Y : \mathcal{C}_Y \to (Y, y)$$

satisfying conditions (i)$-$(iii) of Lemma 2. Via $\tau : \mathcal{C}_Y \to \mathcal{E}$, μ naturally induces a map

$$\mu_Y : \mathcal{C}_Y \to \mathbf{P}^r.$$

The family $(\pi_Y : \mathcal{C}_Y \to (Y, y), \{p_i\}, \mu_Y)$ is certainly an extension of the family over $Y \setminus \{y\}$ determined by the γ pull-back of the stable family (20). The special fiber is a map of a pointed quasi-stable curve to $\mathbf{P}^r$. Unfortunately, the special fiber may not be stable. A stable family of maps is produced in two steps. First, unmarked, μ_Y-collapsed, -1-curves in the special fiber are sequentially blow-down. A multiple of the line bundle

$$\omega_{\pi_Y}(\sum_i p_i) \otimes \mu_Y^*(\mathcal{O}_{\mathbf{P}^r}(3)) \tag{21}$$

is then π_Y- relatively basepoint free. Second, as in [Kn], the relative morphism determined by a power of the line bundle (21) blows-down the remaining destabilizing $\mathbf{P}^1$'s to yield a stable extension over (Y, y). □

Lemma 2. *Let $\pi_X : \mathcal{S}_X \to (X, x)$ be a flat, projective family of curves with l sections $s_1, \ldots, s_l$ satisfying the following condition: $\forall \xi \neq x$, $\pi^{-1}(\xi) = \mathcal{C}_\xi$ is a projective nonsingular curve with l distinct marked points $s_1(\xi), \ldots, s_l(\xi)$. There exists a base change $\gamma : (Y, y) \to (X, x)$ étale except possibly at y with a family of*

l-pointed curves $\pi_Y : \mathcal{S}_Y \to (Y, y)$ *and a diagram:*

$$\begin{array}{ccc} \mathcal{S}_Y & \xrightarrow{\tau} & \mathcal{S}_X \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi_X} \\ (Y, y) & \xrightarrow{\gamma} & (X, x) \end{array}$$

satisfying the following properties:

(i) $\mathcal{S}_Y$ *is a nonsingular surface.* $\pi_Y : \mathcal{S}_Y \to (Y, y)$ *is a flat, projective family of* l*-pointed quasi-stable curves.*

(ii) *For each marking* $1 \leq i \leq l$, $\tau \circ s_i = s_i \circ \gamma$.

(iii) *Over* $W = Y \setminus \{y\}$, *there is isomorphism* $\mathcal{S}_W \xrightarrow{\sim} \gamma_W^*(\mathcal{S}_U)$, *where* $U = X \setminus \{x\}$. *The morphism* $\tau|_{\mathcal{S}_W}$ *is the composition*

$$\mathcal{S}_W \xrightarrow{\sim} \gamma_W^*(\mathcal{S}_U) \to \mathcal{S}_U$$

where the second map is the natural projection.

Proof. The method is by standard semi-stable reduction (cf. [K-K-M], [Ha]). First, the singularities of $\mathcal{S}_X$ are resolved. Note that all singularities lie in the special fiber. Next, the surface $\mathcal{S}_X$ is blown-up sufficiently to ensure the reduced scheme supported on the special fiber has normal crossing singularities in $\mathcal{S}_X$. The required blow-ups have point centers in the special fiber. Finally, the resulting surface is blown-up further (at points in the special fiber) to ensure the marking sections $s_1, \ldots, s_l$ do not intersect each other and do not pass through nodes of the reduced scheme supported on the special fiber. Let $\hat{\pi} : \hat{\mathcal{S}}_X \to (X, x)$ be the resulting nonsingular surface. The singularities of the morphism $\hat{\pi}$ are locally of the form $z_1^\alpha z_2^\beta = t$ where z_1, z_2 are coordinates on $\hat{\mathcal{S}}_X$ and t is a coordinate on X. Let $\{\alpha_j, \beta_j\}$ be the set of exponents that occur at the singularities of $\hat{\pi}$. Let $\gamma : (Y, y) \to (X, x)$ be a base change whose ramification index over x is divisible by all α_j and β_j. Let $\mathcal{S}_Y$ be the normalization of $\gamma^*(\hat{\mathcal{S}}_X)$. A straightforward local analysis shows the family $\pi_Y : \mathcal{S}_Y \to (Y, y)$ has an l-pointed, reduced, nodal special fiber. The surface $\mathcal{S}_Y$ has singularities of the local form $z_1 z_2 - t^k$ in the special fiber. Blowing-up $\mathcal{S}_Y$ yields a nonsingular surface with the required properties (i)−(iii). □

By the valuative criterion, Propositions 5 and 6 prove $\overline{M}_{g,n}(\mathbf{P}^r, d)$ is a separated and proper complex algebraic scheme.

4.3. Projectivity. The projectivity of the proper schemes $\overline{M}_{g,n}(\mathbf{P}^r, d)$ is established here by a method due to J. Kollár ([Ko1]). Proofs of the projectivity of $\overline{M}_{g,n}(\mathbf{P}^r, \beta)$ can also be found in [A] and [C]. Kollár constructs ample line bundles on proper spaces via sufficiently nontrivial quotients of semipositive vector bundles. A vector bundle E on an algebraic scheme S is *semipositive* if for every morphism of a projective curve $f : C \to S$, every quotient line bundle of $f^*(E)$ has nonnegative degree on C.

The first step is a semipositivity Lemma. Let

$$\mathcal{F} = (\pi : \mathcal{C} \to S,\ \{p_i\},\ \mu) \tag{22}$$

be a stable family of maps over S to $\mathbf{P}^r$. Let

$$E_k(\pi) = \pi_* \left(\omega_\pi^k (\sum_{i=1}^n k p_i) \otimes \mu^*(\mathcal{O}(3k)) \right).$$

LEMMA 3. *$E_k(\pi)$ is a semipositive vector bundle on S for $k \geq 2$.*

Proof. A slight perturbation of the arguments in [Ko1] is required. It suffices to prove semipositivity in case the base is a nonsingular curve X. Let $\gamma : Y \to X$ be a flat base change. By map stability, Serre duality, and the base change theorems, it follows (for $k \geq 2$) E_k commutes with pull-back:

$$E_k(\pi_Y) \cong \gamma^*(E_k(\pi_X))$$

where π_Y is the pull-back family over Y. It therefore suffices to prove semipositivity after base change.

Using the methods of section 4.2, it can be assumed (after base change) that $\mathcal{F}$ is a union of component stable families $\mathcal{F}_j = (\pi_j : \mathcal{C}_j \to X,\ \{p_i^j\}, \{p_i^c\},\ \mu_j)$ where π_j is family of stable maps and the generic element of $\mathcal{F}_j$ is a map of an *irreducible*, projective, nodal curve. The notation introduced in the proof of Proposition 6 is employed. After further base change and normalization of $\mathcal{F}_j$, it can be assumed that

$$\tilde{\mathcal{F}}_j = (\tilde{\pi}_j : \tilde{\mathcal{C}}_j \to X,\ \{p_i^j\}, \{p_i^c\}, \{p_i^n\},\ \tilde{\mu}_j) \tag{23}$$

is a family of stable maps where the generic element is a map of an irreducible, projective, *nonsingular* curve.

A semipositivity result for the family $\tilde{\mathcal{F}}_j$ is first established. Let H_1, H_2, $H_3 \subset \mathbf{P}^r$ be general hyperplanes. After base change, it can be assumed $\tilde{\mu}_j^*(H_l)$ is a union of d reduced sections for each l. These $3d$ sections are distinct from the sections $\{p_i^j\}$, $\{p_i^c\}$, $\{p_i^n\}$. Therefore,

$$\omega_{\tilde{\pi}_j}^k\big(\sum k p_i^j + \sum (k-1)p_i^c + \sum (k-1)p_i^n\big) \otimes \tilde{\mu}_j^*(\mathcal{O}_{\mathbf{P}^r}(3k)) \xrightarrow{\sim} \omega_{\tilde{\pi}_j}^k\big(\sum \alpha_q X_q\big) \tag{24}$$

where X_q are distinct sections of $\tilde{\pi}_j$ and $\alpha_q \leq k$. The surface $\tilde{\mathcal{C}}_j$ has finitely many singularities of the form $z_1 z_2 - t^\alpha$. These singularities are resolved by blow-up,

$$\tau : \mathcal{S}_j \to \tilde{\mathcal{C}}_j.$$

Since the relative dualizing sheaf of the family $\mathcal{S}_j$ is trivial on the exceptional $\mathbf{P}^1$'s of τ, Lemma 4 below can be applied to deduce the semipositivity of $F_k(\tilde{\pi}_j)$ for $k \geq 2$ where

$$F_k(\tilde{\pi}_j) = \tilde{\pi}_{j*}\Big(\omega_{\tilde{\pi}_j}^k\big(\sum k p_i^j + \sum (k-1)p_i^c + \sum (k-1)p_i^n\big) \otimes \mu^*(\mathcal{O}(3k))\Big).$$

For $k \geq 2$, the restriction of the line bundle (24) to a fiber of $\tilde{\mathcal{C}}_j$ is equal to

$$\omega \otimes \omega^{k-1}\big(\sum (k-1)p_i^j + \sum (k-1)p_i^c + \sum (k-1)p_i^n\big) \otimes \tilde{\mu}_j^*(\mathcal{O}_{\mathbf{P}^r}(3k-3)) \otimes \mu_j^*(\mathcal{O}_{\mathbf{P}^r}(3))\big(\sum_i p_i^j\big) \tag{25}$$

where ω is the dualizing sheaf of the fiber. By stability for the family $\tilde{\mathcal{F}}_j$, the product of the middle two factors in (25) is ample for $k \geq 2$. The last factor in (25) is certainly of non-negative degree. By Serre duality, for $k \geq 2$,

$$R^1\tilde{\pi}_{j*}\Big(\omega_{\tilde{\pi}_j}^k\big(\sum k p_i^j + \sum (k-1)p_i^c + \sum (k-1)p_i^n\big) \otimes \mu^*(\mathcal{O}(3k))\Big) = 0. \tag{26}$$

The semipositivity of $E_k(\pi)$ will be obtained from the semipositivity of $F_k(\tilde{\pi}_j)$. The $(k-1)$-multiplicities will naturally arise in considering dualizing sheaves on nodal and reducible curves.

Let $\tilde{\pi}_{\cup_j} : \bigcup_j \tilde{\mathcal{C}}_j \to X$ be the disjoint union of the families $\tilde{\mathcal{F}}_j$. There is natural morphism from the disjoint union to $\mathcal{C}$

$$\rho : \bigcup_j \tilde{\mathcal{C}}_j \to \mathcal{C}$$

obtained by identifying nodal marked points and gluing components along intersection marked points. Consider the natural sequence of sheaves on $\mathcal{C}$:

$$0 \to \rho_*(\omega_{\tilde{\pi}_{\cup_j}}) \to \omega_\pi \to K \to 0. \tag{27}$$

The quotient K is easily identified as $\bigoplus_{p_i^c, p_i^n} \mathcal{O}_p$ where the sum is over all nodal and component intersection sections of the family $\mathcal{F}$. Tensoring (27) with the line bundle $\omega_\pi^{k-1}(\sum kp_i) \otimes \mu^*(\mathcal{O}_{\mathbf{P}(V)}(3k))$ yields the exact sequence:

$$0 \to \rho_*\left(\omega^k_{\tilde{\pi}_{\cup_j}}(\sum kp_i + \sum (k-1)p_i^c + \sum (k-1)p_i^n) \otimes \mu^*(\mathcal{O}_{\mathbf{P}(V)}(3k))\right) \to$$

$$\omega_\pi^k(\sum_i kp_i) \otimes \mu^*(\mathcal{O}_{\mathbf{P}(V)}(3k)) \to \bigoplus_{p_i^c, p_i^n} \mathcal{O}_p \otimes \mu^*(\mathcal{O}_{\mathbf{P}(V)}(3k)) \to 0.$$

Certainly $\tilde{\pi}_{\cup_j *} = \pi_* \rho_*$. Note the vanishing of R^1 determined in (26). These facts imply the π direct image of the above sequence on $\mathcal{C}$ yields an exact sequence on X:

$$0 \to \bigoplus_j F_k(\tilde{\pi}_j) \to E_k(\pi) \to \bigoplus_{p_i^c, p_i^n} \mathcal{O}_X \otimes \mu^*(\mathcal{O}_{\mathbf{P}(V)}(3k)) \to 0.$$

Finally, since an extension of semipositive bundles is semipositive ([Ko1]), $E_k(\pi)$ is semipositive. □

LEMMA 4. *Let $\pi : \mathcal{S} \to X$ be a map from a nonsingular projective surface to a nonsingular curve. Assume the general fiber of π is nonsingular. Let X_q be a set of distinct sections of π. Then*

$$\pi_*(\omega^k_{\mathcal{S}/X}(\sum \alpha_q X_q))$$

is semipositive provided $k \geq 2$ and $\alpha_q \leq k$ for all q.

Proof. This is precisely Proposition 4.7 of [Ko1]. □

The second step is the construction of a non-trivial quotient. Let $\mathcal{F}$ be the family (22). Let $\mathbf{P}(E_k^*)$ be the projective bundle over S obtained from the subspace projectivization of E_k^*. The condition of stability implies there is a canonical S-embedding $\iota : \mathcal{C} \to \mathbf{P}(E_f^*)$ for some $f = f(d, g, n, r)$ (see section 2.3). The morphism μ then yields a canonical S-embedding:

$$\gamma : \mathcal{C} \to \mathbf{P}(E_f^*) \times_{\mathbb{C}} \mathbf{P}^r.$$

The n sections $\{p_i\}$ yield n sections $\{(\iota \circ p_i, \mu \circ p_i)\}$ of $\mathbf{P}(E_f^*) \times \mathbf{P}^r$ over S. Let S_i denote the subscheme of $\mathbf{P}(E_f^*) \times \mathbf{P}^r$ defined by the i^{th} section. Denote the projection of $\mathbf{P}(E_f^*) \times \mathbf{P}^r$ to S also by π. Let $\mathcal{M} = \mathcal{O}_{\mathbf{P}(E_f^*)}(1) \otimes \mathcal{O}_{\mathbf{P}^r}(1)$. $\mathcal{M}$ is an π-relatively ample line bundle. Note $\pi_*(\mathcal{M}^l) \cong \mathrm{Sym}^l(E_f) \otimes \mathrm{Sym}^l(\mathbb{C}^{r+1})$. By the stability of semipositivity under symmetric and tensor products ([Ko1]) and

Lemma (3), $\pi_*(\mathcal{M}^l)$ is semipositive. Fix a choice of l (depending only on d, g, n, and r) large enough to ensure

$$\pi_*(\mathcal{M}^l) \oplus \bigoplus_1^n \pi_*(\mathcal{M}^l) \to \pi_*(\mathcal{M}^l \otimes \mathcal{O}_\mathcal{C}) \oplus \bigoplus_1^n \pi_*(\mathcal{M}^l \otimes \mathcal{O}_{S_i}) \to 0. \tag{28}$$

Such a choice of l is possible by the boundedness established in section 2.3. Let Q be the quotient in (28). By uniform cohomology vanishing, the quotient Q is a vector bundle for large l.

The quotient (28) is nontrivial in the following sense. Let $G = GL$ be the structure group of the bundle E_f. G is naturally the structure group of $\pi_*(\mathcal{M}^l)$. Let W be the G-representation inducing the bundle $\pi_*(\mathcal{M}^l) \oplus \bigoplus_1^n \pi_*(\mathcal{M}^l)$. Let q be the rank of the quotient bundle of (28). The quotient sequence (28) yields a set theoretic classifying map to the Grassmannian:

$$\rho : S \to \mathbf{Gr}(q, W^*)/G.$$

Lemma 5. *There exists a set theoretic injection*

$$\delta : \overline{M}_{g,n}(\mathbf{P}^r, d) \to \mathbf{Gr}(q, W^*)/G.$$

Let $\lambda : S \to \overline{M}_{g,n}(\mathbf{P}^r, d)$ be the map induced by the stable family (22). There is a (set theoretic) factorization $\rho = \delta \circ \lambda$.

Proof. For large l, the sequence (28) is equivalent to the data of a Hilbert point in J (see section 2.3). Since the G orbits of J are exactly the stable maps, the Lemma follows. □

Lemma 6. *A stable map has a finite number of automorphisms.*

Proof. As simple consequence of the definition of stability, there are no infinitesimal automorphisms. The total number is therefore finite. □

Suppose the map to moduli $\lambda : S \to \overline{M}_{g,n}(\mathbf{P}^r, d)$ is a generically finite algebraic morphism. Then, in the terminology of [Ko1], Lemmas 5 and 6 show the classifying map ρ is *finite* on an open set of S.

Proposition 7. *(Lemma 3.13, [Ko1]) Let the base S of (22) be a normal projective variety. Suppose the classifying map is finite on an open set of S. Then, the top self-intersection number of $Det(Q)$ on S is positive.*

If $\overline{M}_{g,n}(\mathbf{P}^r, d)$ were a fine moduli space equipped with a universal family, $\mathrm{Det}(Q)$ would be well defined and ample (by Proposition 7 and the Nakai-Moishezon criterion) on $\overline{M}_{g,n}(\mathbf{P}^r, d)$. Since $\overline{M}_{0,n}(\mathbf{P}^r, d)$ is expressed locally as a quotient of a fine moduli space by a finite group, it is easily seen $\mathrm{Det}(Q)^k$ is a well defined line bundle on $\overline{M}_{0,n}(\mathbf{P}^r, d)$ for sufficiently large k. The exponent k is taken to trivialize the $\mathbb{C}^*$-representations at the fixed points. In the higher genus case, $\mathrm{Det}(Q)$ is a well defined line bundle on the Hilbert scheme J or the stack. Since the moduli problem has finite automorphisms, $\mathrm{Det}(Q)^k$ is well defined on the coarse moduli space for some k.

Since the moduli spaces $\overline{M}_{g,n}(\mathbf{P}^r, d)$ are not fine, subvarieties are not equipped with stable families. Proposition (7) and the Nakai-Moishezon criterion do not directly establish the ampleness of $\mathrm{Det}(Q)^k$. An alternative approach (due to J.

Kollár) is followed. Recall the Hilbert scheme J (of section 2.3) is equipped with a universal family and, therefore, a canonical map

$$J \to \overline{M}_{g,n}(\mathbf{P}^r, d).$$

Let $X \subset \overline{M}_{g,n}(\mathbf{P}^r, d)$ be a subvariety. Using J and the finite automorphism property of a stable map, a morphism $Y \to X$ of algebraic schemes can be constructed satisfying

(i) $Y \to X$ is finite and surjective.
(ii) Y is equipped with a stable family of maps such that $Y \to X$ is the corresponding morphism to moduli.

The existence of $Y \to X$ is exactly the conclusion of Proposition 2.7 in [Ko1] under slightly different assumptions. Nevertheless, the argument is valid in the present setting. The construction of Y is subtle. First Y is constructed as an algebraic space. Then, a Lemma of Artin is used to find an algebraic scheme Y. Since Y has a universal family, Proposition 7 implies $\mathrm{Det}(Q)^k$ has positive top intersection on Y and therefore on X. The Nakai-Moishezon criterion can be applied to conclude the ampleness of $\mathrm{Det}(Q)^k$ on $\overline{M}_{g,n}(\mathbf{P}^r, d)$.

4.4. Automorphisms. We use the notation of sections 3.2 and 4.1. In the genus 0 case, $\overline{M}(\overline{t})$ is nonsingular. Therefore, the space $\overline{M}_{0,n}(\mathbf{P}^r, d)$ is locally a quotient of a nonsingular variety by a finite group.

LEMMA 7. *Let $\xi \in \overline{M}(\overline{t})$ be a point at which the $G_{d,r}$ action is not free. Then ξ corresponds to a stable map with nontrivial automorphisms.*

Proof. $G_{d,r}$ acts by isomorphism on the stable maps of the universal family over $\overline{M}(\overline{t})$. The $G_{d,r}$ action is not free at $\xi \in \overline{M}(\overline{t})$ if and only if there exists a $1 \neq \gamma \in G_{d,r}$ fixing ξ. The element γ induces an automorphism of the map corresponding to ξ. The automorphism is nontrivial on the marked points $\{q_{i,j}\}$. □

Over the automorphism-free locus, the $G_{d,r}$-action on $\overline{M}(\overline{t})$ (and on the universal family over $\overline{M}(\overline{t})$) is free. Hence, the quotient over the automorphism-free locus is nonsingular and equipped with a universal family. A universal family of over the nonsingular quasi-projective variety $\overline{M}_{0,n}(\mathbf{P}^r, d)$ is obtained by patching. Theorems 1 and 2 have been established in the case $X \cong \mathbf{P}^r$.

5. The construction of $\overline{M}_{g,n}(X, \beta)$

5.1. Proof of Theorem 1. Let X be a projective algebraic variety. Existence of the coarse moduli space $\overline{M}_{g,n}(X, \beta)$ is established via a projective embedding $\iota : X \hookrightarrow \mathbf{P}^r$. Let $\iota_*(\beta)$ be d times the class of a line in $\mathbf{P}^r$.

LEMMA 8. *There exists a natural closed subscheme*

$$\overline{M}_{g,n}(X, \beta, \overline{t}) \subset \overline{M}_{g,n}(\mathbf{P}^r, d, \overline{t})$$

satisfying the following property. Let $(\pi : \mathcal{C} \to S, \{p_i\}, \{q_{i,j}\}, \mu)$ be a $\overline{t}$-rigid stable family of genus g, n-pointed, degree d maps to $\mathbf{P}^r$. Then, the natural morphism $S \to \overline{M}_{g,n}(\mathbf{P}^r, d, \overline{t})$ factors through $\overline{M}_{g,n}(X, \beta, \overline{t})$ if and only if μ factors through ι and each geometric fiber of π is a map to X representing the homology class $\beta \in A_1 X$.

Proof. The Lemma is proved in case $g = 0$. If $g > 0$, then $\overline{M}_{g,n}(\mathbf{P}^r, d, \overline{t})$ is not a fine moduli space and the argument is more technical.

Let

$$(\pi_M : \mathcal{U} \to \overline{M}_{0,n}(\mathbf{P}^r, d, \overline{t}), \{p_i\}, \{q_{i,j}\}, \mu)$$

be the universal family over $\overline{M}_{0,n}(\mathbf{P}^r, d, \overline{t})$. On a genus 0 curve, any vector bundle generated by global sections has no higher cohomology. Therefore, by this cohomology vanishing and the base change theorems, $\pi_{M*}\mu^*(\mathcal{O}_{\mathbf{P}^r}(k))$ is a vector bundle for all $k > 0$. (This argument must be modified in the $g > 0$ case since $\pi_{M*}\mu^*(\mathcal{O}_{\mathbf{P}^r}(k))$ need not be a vector bundle even on the Hilbert scheme J or the stack. Nevertheless, it is not hard to define the closed subscheme determined by X on the Hilbert scheme J or the stack and then descend it to the coarse moduli space.) Let $\mathcal{I}_X$ be the ideal sheaf of $X \subset \mathbf{P}^r$. Let $I_X(k) = H^0(\mathbf{P}^r, \mathcal{I}_X(k))$. Let $l >> 0$ be selected so that $\mathcal{I}_X(l)$ is generated by the global sections $I_X(l)$. These sections $I_X(l)$ yield sections of the vector bundle $\pi_{M*}\mu^*(\mathcal{O}_{\mathbf{P}^r}(k))$. Let $Z \subset \overline{M}_{0,n}(\mathbf{P}^r, d, \overline{t})$ be the scheme-theoretic zero locus of these sections. The restriction of μ to $\pi_M^{-1}(Z)$ factors though ι. Since Z is an algebraic scheme, Z is a finite union of disjoint connected components. The homology class in $A_1(X) = H_2(X, \mathbb{Z})$ represented by a map with moduli point in Z is a deformation invariant of the map. Therefore, the represented homology class is constant on each connected component of Z. Let $Z_\beta \subset Z$ be the union of components of Z which consist of maps representing the class $\beta \in A_1 X$. Let $\overline{M}_{0,n}(X, \beta, \overline{t}) = Z_\beta$. The required properties are easily established. □

By the functorial property, $\overline{M}_{g,n}(X, \beta, \overline{t})$ is invariant under the $G_{d,r} \cong \mathfrak{S}_d \times \cdots \times \mathfrak{S}_d$ action on $\overline{M}_{g,n}(\mathbf{P}^r, d, \overline{t})$. The quotient

$$\overline{M}_{g,n}(X, \beta, \overline{t})/G_{d,r}$$

is an open set of $\overline{M}_{g,n}(X, \beta)$. A patching argument identical to the patching argument of section 4.1 yields a construction of $\overline{M}_{g,n}(X, \beta)$ as a closed scheme of $\overline{M}_{g,n}(\mathbf{P}^r, d)$. The functorial property of $\overline{M}_{g,n}(X, \beta)$ shows the construction is independent of the projective embedding of X. Projectivity of $\overline{M}_{g,n}(X, \beta)$ is obtained from the projectivity of $\overline{M}_{g,n}(\mathbf{P}^r, d)$. This completes the proof of Theorem 1.

5.2. Proof of Theorem 2. A deformation study is needed to establish Theorem 2. Let $g = 0$ and let X be a projective, nonsingular, convex variety. Theorem 2 is certainly true in case $\beta = 0$: $\overline{M}_{0,n}(X, 0) = \overline{M}_{0,n} \times X$. We assume $\beta \neq 0$. By the functorial property, the Zariski tangent space to the scheme $\overline{M}_{0,n}(X, \beta, \overline{t})$ at the point $(C, \{p_i\}, \{q_{i,j}\}, \mu : C \to X)$ is canonically isomorphic to the space of first order deformations of the pointed $\overline{t}$-stable map $(C, \{p_i\}, \{q_{i,j}\}, \mu : C \to X)$. The later deformation space corresponds bijectively to the space of first order deformations of the pointed stable map $(C, \{p_i\}, \mu : C \to X)$.

Let $\mathrm{Def}(\mu)$ denote the space of first order deformations of the pointed stable map $(C, \{p_i\}, \mu : C \to X)$. Consider first the case in which $C \cong \mathbf{P}^1$. Let $\mathrm{Def}_R(\mu)$ be the space of first order deformations of $(C, \{p_i\}, \mu : C \to X)$ with C held rigid. There is an natural exact sequence:

$$0 \to H^0(C, T_C) \to \mathrm{Def}_R(\mu) \to \mathrm{Def}(\mu) \to 0.$$

(The assumption $\beta \neq 0$ implies the left map is injective.) Let $\mathrm{Hom}(C, X)$ be the quasi-projective scheme of morphisms from C to X representing the class β.

$\mathrm{Hom}(C,X)$ is an open subscheme of the Hilbert scheme of graphs in $C \times X$. The Zariski tangent space to $\mathrm{Hom}(C,X)$ is naturally identified:

$$T_{\mathrm{Hom}(C,X)}([\mu]) \cong H^0(C, \mu^* T_X)$$

(see [Ko2]). There is an exact sequence:

$$0 \to \mathrm{Ker} \to \mathrm{Def}_R(\mu) \to H^0(C, \mu^* T_X) \to 0$$

where Ker corresponds to the deformations of the markings. Therefore, $\dim_{\mathbb{C}}\mathrm{Ker} = n$. Since X is convex, the above sequences suffice to compute the dimension of $\mathrm{Def}(\mu)$

$$\dim_{\mathbb{C}}\mathrm{Def}(\mu) = \dim(X) + \int_\beta c_1(T_X) + n - 3.$$

The dimension of the tangent space to $\overline{M}_{0,n}(X,\beta,\overline{t})$ is established in case $C \cong \mathbf{P}^1$.

Before proceeding further, the following deformation result is needed. A proof can be found in [Ko2].

LEMMA 9. *Let $\mathcal{C}/S$ and $\mathcal{X}/S$ be flat, projective schemes over S. Let $s \in S$ be geometric point. Let C_s, X_s be the fibers over s and let $f : C_s \to X_s$ be a morphism. Assume the following conditions are satisfied:*

(i) *C_s has no embedded points.*
(ii) *X_s is nonsingular.*
(iii) *S is equidimensional at s.*

Then, the dimension of every component of the quasi-projective variety $Hom_S(\mathcal{C},\mathcal{X})$ at the point $[f]$ is at least

$$\dim_{\mathbb{C}} H^0(C_s, f^* T_{X_s}) - \dim_{\mathbb{C}} H^1(C_s, f^* T_{X_s}) + \dim_s S.$$

Again, let $(C \cong \mathbf{P}^1, \{p_i\}, \{q_{i,j}\}, \mu : C \to X)$ correspond to a point of the space $\overline{M}_{0,n}(X,\beta,\overline{t})$. By Lemma 9 and the convexity of X, every component of $\mathrm{Hom}(C,X)$ at $[\mu]$ has dimension at least $\dim_{\mathbb{C}} H^0(C, \mu^* T_X)$. Therefore, every component of $\overline{M}_{0,n}(X,\beta,\overline{t})$ at $[\mu]$ has dimension at least $\dim(X) + \int_\beta c_1(T_X) + n - 3$. By the previous tangent space computation, it follows $[\mu]$ is a nonsingular point of $\overline{M}_{0,n}(X,\beta,\overline{t})$. Before attacking the reducible case, a lemma is required.

LEMMA 10. *Let X be a nonsingular, projective, convex space. Let $\mu : C \to X$ be a morphism of a projective, connected, reduced, nodal curve of arithmetic genus 0 to X. Then,*

$$H^1(C, \mu^* T_X) = 0. \tag{29}$$

and $\mu^* T_X$ is generated by global sections on C.

Proof. Let $E \subset C$ be an irreducible component of C; $E \cong \mathbf{P}^1$. Let

$$\mu^* T_X|_E \cong \bigoplus \mathcal{O}_{\mathbf{P}^1}(\alpha_i).$$

Suppose there exists $\alpha_i < 0$. The composition of a rational double cover of E with μ would then violate the convexity of X . It follows that:

$$\forall i, \quad \alpha_i \geq 0. \tag{30}$$

We will prove the following statement by induction on the number of components of C:

$$H^1(C, \mu^* T_X \otimes \mathcal{O}_C(-p)) = 0 \tag{31}$$

for any nonsingular point $p \in C$. Equation (31) is true by condition (30) when $C \cong \mathbf{P}^1$ is irreducible. Assume now C is reducible and $p \in E \cong \mathbf{P}^1$. Let $C = C' \cup E$; let $p' = C' \cap E$. There is a component sequence:

$$0 \to \mu^* T_X|_{C'} \otimes \mathcal{O}_{C'}(-p') \to \mu^* T_X \otimes \mathcal{O}_C(-p) \to \mu^* T_X|_E \otimes \mathcal{O}_E(-p) \to 0.$$

Equation (31) now follows from the inductive assumptions on C' and E.

We now prove $H^1(C, \mu^* T_X) = 0$. If $C \cong \mathbf{P}^1$, then the lemma is established by condition (30). Assume now $C = C' \cup E$ where $E \cong \mathbf{P}^1$. There is a component sequence

$$0 \to \mu^* T_X|_{C'} \otimes \mathcal{O}_{C'}(-p') \to \mu^* T_X \to \mu^* T_X|_E \to 0. \tag{32}$$

Equation (29) now follows from (31) applied to C'.

Finally, an analysis of sequence (32) also yields the global generation result. $\mu^* T_X|_E$ is generated by global sections by (30). Sequence (32) is exact on global sections by (31). Hence $\mu^* T_X$ is generated by global sections on E. But, every point of C lies on some component $E \cong \mathbf{P}^1$. □

In section 8, the following related Lemma will be required:

LEMMA 11. *Let $\mu : \mathbf{P}^1 \to X$ be a non-constant morphism to a nonsingular, projective, convex space X. Then $\int_{\mu_*[\mathbf{P}^1]} c_1(T_X) \geq 2$.*

Proof. Since μ is non-constant, the differential

$$d\mu : T_{\mathbf{P}^1} \to \mu^*(T_X)$$

is nonzero. Let $s \in H^0(\mathbf{P}^1, T_{\mathbf{P}^1})$ be a vector field with two distinct zeros $p_1, p_2 \in \mathbf{P}^1$. Then, $d\mu(s) \in H^0(\mathbf{P}^1, \mu^*(T_x)) \neq 0$ and $d\mu(s)$ vanishes (at least) at p_1 and p_2. By the proof of Lemma 10, $\mu^*(T_X) \cong \bigoplus \mathcal{O}_{\mathbf{P}^1}(\alpha_i)$ where $\alpha_i \geq 0$ for all i. The existence of $d\mu(s)$ implies that $\alpha_j \geq 2$ for some j. □

Let C now be a reducible curve. C must be a tree of $\mathbf{P}^1$'s. Let q be the number of nodes of C. Again, let $\mathrm{Def}(\mu)$ be the first order deformation space of the pointed stable map μ. The *dual graph* of a pointed curve C of arithmetic genus 0 consists of vertices and edges corresponding bijectively to the irreducible components and nodes of C respectively. The *valence* of a vertex in the dual graph is the numbers of edges incident at that vertex. Let $\mathrm{Def}_G(\mu) \subset \mathrm{Def}(\mu)$ be the first order deformation space of the stable map μ preserving the dual graph. $\mathrm{Def}_G(\mu)$ is a linear subspace of codimension at most q. Let $\mathrm{Def}_G(C)$ be the space of first order deformations of the curve C which preserve the dual graph. A simple calculation yields

$$\dim_{\mathbb{C}} \mathrm{Def}_G(C) = \sum_{|\nu| \geq 4} |\nu| - 3$$

where the sum is taken over vertices ν of the dual graph of valence at least 4.

The natural linear map $\mathrm{Def}_G(\mu) \to \mathrm{Def}_G(C)$ is now analyzed. Let S be the nonsingular universal base space of deformations of C preserving the dual graph. Let $\mathcal{C}$ be the universal deformation over S. Let $\mathcal{X} = X \times S$. Let $s_0 \in S$ correspond to C. By Lemmas 9 and 10, every component of $\mathrm{Hom}_S(\mathcal{C}, \mathcal{X})$ has dimension at least $\dim(X) + \int_\beta c_1(T_X) + \dim(S)$. The tangent space to the fiber of $\mathrm{Hom}_S(\mathcal{C}, \mathcal{X})$ over s_o at $[\mu]$ is canonically $H^0(C, \mu^* T_X)$. The latter space has dimension $\dim(X) + \int_\beta c_1(T_X)$. Hence, $\mathrm{Hom}_S(\mathcal{C}, \mathcal{X})$ is nonsingular at $[\mu]$ of dimension

$\dim(X) + \int_\beta c_1(T_X) + \dim(S)$ and the projection morphism to S is smooth at $[\mu]$. Therefore, $\mathrm{Def}_G(\mu) \to \mathrm{Def}_G(C)$ is surjective.

The above definitions and results yield a natural exact sequence:

$$0 \to \mathrm{Def}_C(\mu) \to \mathrm{Def}_G(\mu) \to \mathrm{Def}_G(C) \to 0$$

where $\mathrm{Def}_C(\mu)$ is the space of first order deformations of the stable map μ which restrict to the trivial deformation of C. As in the case where $C \cong \mathbf{P}^1$, $\mathrm{Def}_C(\mu)$ differs from $\mathrm{Def}_R(\mu)$ only by the tangent fields obtained from automorphisms:

$$0 \to H^0(C, T_C^{auto}) \to \mathrm{Def}_R(\mu) \to \mathrm{Def}_C(\mu) \to 0.$$

$H^0(C, T_C^{auto})$ is the space of tangent fields on the components of C that vanish at all the nodes of C. Note $H^0(C, T_C^{auto}) = \sum_{|\nu|\leq 3} 3 - |\nu|$. Finally, there is an exact sequence containing $\mathrm{Def}_R(\mu)$ and the tangent space to $\mathrm{Hom}(C, X)$:

$$0 \to \mathrm{Ker} \to \mathrm{Def}_R(\mu) \to H^0(C, \mu^* T_X) \to 0.$$

From these exact sequences, Lemma 10, and some arithmetic, it follows

$$\dim_{\mathbb{C}} \mathrm{Def}_G(\mu) = \dim(X) + \int_\beta c_1(T_X) + n - 3 - q. \tag{33}$$

Let $\mathcal{C}$ be a smoothing of the reducible curve C over a base S and let $\mathcal{X} = X \times S$. A simple application of Lemma 9 shows that $[\mu] \in \overline{M}_{0,n}(X, \beta, \overline{t})$ lies in the closure of the locus of maps with irreducible domains. Since the irreducible domain locus is pure dimensional of dimension $\dim(X) + \int_\beta c_1(T_X) + n - 3$,

$$\dim_{\mathbb{C}} \mathrm{Def}(\mu) \geq \dim(X) + \int_\beta c_1(T_X) + n - 3. \tag{34}$$

It follows from (33) and (34) that $\mathrm{Def}_G(\mu)$ is of maximal codimension q in $\mathrm{Def}(\mu)$ and that the inequality in (34) is an equality. Since $\mathrm{Def}(\mu)$ is of dimension $\dim(X) + \int_\beta c_1(T_X) + n - 3$, $[\mu]$ is a nonsingular point of $\overline{M}_{0,n}(X, \beta, \overline{t})$. Since $\overline{M}_{0,n}(X, \beta, \overline{t})$ is nonsingular of pure dimension $\dim(X) + \int_\beta c_1(T_X) + n - 3$, parts (i) and (ii) of Theorem 2 are established. Part (iii) follows from the corresponding result in the case $X \cong \mathbf{P}^r$.

6. The boundary of $\overline{M}_{0,n}(X, \beta)$

6.1. Definitions. Let X be nonsingular, projective, and convex. Let the genus $g = 0$. The *boundary* of $\overline{M}_{0,n}(X, \beta)$ is the locus corresponding to reducible domain curves. Boundary properties of the Mumford-Knudsen space $\overline{M}_{0,m}$ (where $m = n + d(r+1)$) are passed to $\overline{M}_{0,n}(\mathbf{P}^r, d)$ by the local quotient construction. The boundary locus of $\overline{M}_{0,m}$ is a divisor with normal crossings. Since $\overline{M}_{0,n}(\mathbf{P}^r, d, \overline{t})$ is a product of $\mathbb{C}^*$ bundles over an open set of $\overline{M}_{0,m}$, the boundary locus of $\overline{M}_{0,n}(\mathbf{P}^r, d, \overline{t})$ is certainly a divisor with normal crossing. $\overline{M}_{0,n}(\mathbf{P}^r, d)$ is locally the $G_{d,r}$-quotient of $\overline{M}_{0,n}(\mathbf{P}^r, d, \overline{t})$. The boundary of $\overline{M}_{0,n}(\mathbf{P}^r, d)$ is therefore a union of subvarieties of pure codimension 1. Over the automorphism-free locus, the boundary of $\overline{M}_{0,n}(\mathbf{P}^r, d)$ is a divisor with normal crossings.

Let X be a nonsingular, projective, convex variety. The corresponding boundary results for $\overline{M}_{0,n}(X, \beta)$ are consequences of the deformation analysis of section

5.2. The boundary locus of $\overline{M}_{0,n}(X,\beta,\overline{t})$ is a divisor with normal crossing singularities. A pointed map $\mu : C \to X$ such that C has q nodes lies in the intersection of q branches of the boundary. The dimension computation

$$\dim_{\mathbb{C}}\mathrm{Def}_G(\mu) = \dim\overline{M}_{0,n}(X,\beta,\overline{t}) - q$$

shows these branches intersect transversally at $[\mu]$. This completes the proof of Theorem 3. In particular $\overline{M}_{0,n}(X,\beta)$ has the same boundary singularity type as $\overline{M}_g$ and $\overline{M}_{g,n}$.

A class $\beta \in H_2(X,\mathbb{Z})$ is *effective* if β is represented by some genus 0 stable map to X. If $n = 0$, the boundary of $\overline{M}_{0,0}(X,\beta)$ decomposes into a union of divisors which are in bijective correspondence with effective partitions $\beta_1 + \beta_2 = \beta$. For general n, the boundary decomposes into a union of divisors in bijective correspondence with data of weighted partitions $(A,B;\beta_1,\beta_2)$ where

(i) $A \cup B$ is a partition of $[n] = \{1,2,\ldots,n\}$.
(ii) $\beta_1 + \beta_2 = \beta$, β_1 and β_2 are effective .
(iii) If $\beta_1 = 0$ (resp. $\beta_2 = 0$), then $|A| \geq 2$ (resp. $|B| \geq 2$).

$D(A,B;\beta_1,\beta_2)$, the divisor corresponding to the data $(A,B;\beta_1,\beta_2)$, is defined to be the locus of maps $\mu : C_A \cup C_B \to X$ satisfying the following conditions:

(a) C is a union of two quasi-stable curves C_A, C_B of genus 0 meeting in a point.
(b) The markings of A (resp. B) lie on C_A (resp. C_B).
(c) The map $\mu_A = \mu|_{C_A}$ (resp. μ_B) represents β_1 (resp. β_2).

The deformation results of section 5 show the locus maps satisfying (a)$-$(c) and $C_A \cong C_B \cong \mathbf{P}^1$ is dense in $D(A,B;\beta_1,\beta_2)$. If $X = \mathbf{P}^r$, then it is easily seen that $D(A,B;\beta_1,\beta_2)$ is irreducible. In general, we do not claim the divisor $D(A,B;\beta_1,\beta_2)$ is irreducible, although that is the case in all examples we have seen.

6.2. Boundary divisors. The boundary component of $\overline{M}_{0,n}$ corresponding to the marking partition $A \cup B = [n]$ is naturally isomorphic (by gluing) to the product

$$\overline{M}_{0,A\cup\{\bullet\}} \times \overline{M}_{0,B\cup\{\bullet\}}.$$

An analogous construction exists for the boundary divisor $D(A,B;\beta_1,\beta_2)$ of the space $\overline{M}_{0,n}(X,\beta)$.

Let $K = D(A,B;\beta_1,\beta_2)$ be a boundary component of $\overline{M}_{0,n}(X,\beta)$. Let $\overline{M}_A = \overline{M}_{0,A\cup\{\bullet\}}(X,\beta_1)$ and $\overline{M}_B = \overline{M}_{0,B\cup\{\bullet\}}(X,\beta_2)$. Let $e_A : \overline{M}_A \to X$ and $e_B : \overline{M}_B \to X$ be the evaluation maps obtained from the additional marking $\bullet$. Let τ_A, τ_B be the projections of $\overline{M}_A \times \overline{M}_B$ to the first and second factors respectively. Let $\tilde{K} = \overline{M}_A \times_X \overline{M}_B$ be the fiber product with respect to the evaluation maps e_A, e_B. $\tilde{K} \subset \overline{M}_A \times_{\mathbb{C}} \overline{M}_B$ is the closed subvariety $(e_A \times_{\mathbb{C}} e_B)^{-1}(\Delta)$ where $\Delta \subset X \times X$ is the diagonal.

Properties of $\tilde{K}$ can be deduced from the local quotient constructions of $\overline{M}_A$ and $\overline{M}_B$. It will be shown that $\tilde{K}$ is a locally normal, projective variety of pure dimension with finite quotient singularities. Let $\overline{M}_A(X,\overline{t}_A)$, $\overline{M}_B(X,\overline{t}_B)$ be the $\overline{t}_A$, $\overline{t}_B$-rigid moduli spaces. $\tilde{K}$ is the $G_A \times G_B$-quotient of the corresponding subvariety

$$\tilde{K}(X,\overline{t}_A,\overline{t}_B) \subset \overline{M}_A(X,\overline{t}_A) \times \overline{M}_B(X,\overline{t}_B),$$

$$\tilde{K}(X,\overline{t}_A,\overline{t}_B) = (e_A \times_{\mathbb{C}} e_B)^{-1}(\Delta).$$

The differential of e_A at a point $[\mu]$ of $\overline{M}_A(X, \overline{t}_A)$ is determined in the following manner. The case in which the domain $C \tilde{=} \mathbf{P}^1$ is irreducible is most straightforward. Then, there are natural linear maps:

$$\mathrm{Def}(\mu) \to H^0(\mu^* T_X / T_C(-p_\bullet)) \to T_X(\mu(p_\bullet)). \tag{35}$$

The first map in (35) is the natural surjection of $\mathrm{Def}(\mu)$ onto the deformation space of the moduli problem obtained by forgetting all the markings except $\bullet$. The natural fiber evaluation $H^0(\mu^* T_X) \to T_X(\mu(p_\bullet))$ is well defined on the space $H^0(\mu^* T_X / T_C(-p_\bullet))$. This is the second map in (35). The composition of maps in (35) is simply the differential of e_A at $[\mu]$. Since $\mu^* T_X$ is generated by global sections by Lemma 10, it follows that the differential of e_A is surjective at $[\mu]$. A similar argument shows the differential of e_A is surjective for each $[\mu] \in \overline{M}_A(X, \overline{t}_A)$. The differential of e_B is therefore also surjective. The surjectivity of the differentials of e_A and e_B imply $\tilde{K}(X, \overline{t}_A, \overline{t}_B)$ is nonsingular. Thus $\tilde{K}$ is a locally normal, projective variety of pure dimension with finite quotient singularities.

By gluing the universal families over $\overline{M}_A(\overline{t}_A)$ and $\overline{M}_B(\overline{t}_B)$ along the markings $\bullet$, a natural family of Kontsevich stable maps exists over $\tilde{K}(X, \overline{t}_A, \overline{t}_B)$. The induced map

$$\tilde{K}(X, \overline{t}_A, \overline{t}_B) \to K$$

is seen to be $G_A \times G_B$ invariant. Therefore, a natural map $\psi : \tilde{K} \to K$ is obtained.

LEMMA 12. *Results on the morphism ψ :*

(i) *If $A \neq \emptyset$ and $B \neq \emptyset$, then $\psi : \tilde{K} \to K$ is an isomorphism.*
(ii) *If $A \neq \emptyset$, or $B \neq \emptyset$, or $\beta_A \neq \beta_B$, then ψ is birational.*
(iii) *If $A = B = \emptyset$ ($n = 0$) and $\beta_A = \beta_B = \beta/2$ then ψ is generically 2 to 1.*

Proof. First part (i) is proven. Let $q_A \in A$ and $q_B \in B$ be fixed markings (whose existence is guaranteed by the assumptions of (i)). Let $\mathcal{L}$ be a very ample line bundle on X against which all degrees of maps are computed. Let d_A, d_B be the degrees of β_A, β_B respectively. Let $K = (A \cup B, \beta_A, \beta_B)$. Let $\mu : C \to \mathbf{P}^r$ corresponds to a moduli point $[\mu] \in K$. Let $C = \bigcup C_i$ be the union of irreducible components. Let $q_A \in C_1$, $q_B \in C_l$ where $1 \neq l$ and let

$$C_1, C_2, \ldots, C_l$$

be the unique minimal path from C_1 to C_l which exists since C is a tree of components. For $1 \leq i \leq l-1$, let $x_i = C_i \cap C_{i+1}$. Each node x_i divides C into two connected curves

$$C = C_{A,i} \cup C_{B,i}$$

labeled by the points q_A, q_B. Let d_i be the degree of μ restricted to $C_{A,i}$. The degrees d_i increase monotonically. Since $[\mu] \in K$, $d_i = d_A$ for some i. Let j be the minimal value satisfying $d_j = d_A$. If $d_{j+1} > d_A$, then $\psi^{-1}[\mu]$ is the unique point determined by cutting at the node x_j. If $d_{j+1} = d_A$, then the subcurve

$$C \setminus (C_{A,j} \cup C_{B,j+1})$$

must contain (by stability) a nonempty set of marked points P_{j+1}. Let k be maximal index satisfying $d_{j+k} = d_A$. The analogously defined marked point sets

$$P_{j+1}, \ldots, P_{j+k}$$

are all nonempty. There must be a index t satisfying $P_{j+t'} \subset A$ for $1 \leq t' \leq t$ and $P_{j+t'} \subset B$ for $t < t' \leq k$. $\psi^{-1}[\mu]$ is then the unique point determined by cutting at the node x_{j+t}. Therefore, ψ is bijective in case A and B are nonempty.

Let $\overline{M}_{0,n}(X, \beta, \overline{t})$ be a locally rigidified moduli space containing the point $[\mu] \in K$. If $|A|, |B| \geq 1$, a similar argument shows the boundary components of $\overline{M}_{0,n}(X, \beta, \overline{t})$ lying over K are *disjoint*. Therefore, K is normal. In case A and B are nonempty, ψ is a bijective morphism of locally normal varieties and hence an isomorphism.

Note, for example, that the component $K = D(\emptyset, \emptyset; 2, 3)$ of $\overline{M}_{0,0}(\mathbf{P}^r, 5)$ is not normal. K intersects itself along the codimension 2 locus of moduli points $[\mu]$ of the form:

$$\mu : C_1 \cup C_2 \cup C_3 \to \mathbf{P}^r$$

with restricted degrees $d_1 = 2$, $d_2 = 1$, $d_3 = 2$. In this case, $\psi : \tilde{K} \to K$ is a normalization.

Parts (ii) and (iii) follow simply from the defining properties (a)$-$(c) of K. □

7. Gromov-Witten invariants

From now on we assume $g = 0$. A basic 4-point linear equivalence of boundary divisors on $\overline{M}_{0,n}(X, \beta)$ is needed.

Proposition 8. *For i, j, k, l distinct in $[n]$, set*

$$D(i, j \mid k, l) = \sum D(A, B; \beta_1, \beta_2),$$

the sum over all partitions such that i and j are in A, k and l are in B, and β_1 and β_2 are effective classes in A_1X such that $\beta_1 + \beta_2 = \beta$. Then we have the linear equivalence of divisors

$$D(i, j \mid k, l) \sim D(i, l \mid j, k)$$

on $\overline{M}_{0,n}(X, \beta)$.

Proof. The proof is obtained by examining the map

$$\overline{M}_{0,n}(X, \beta) \to \overline{M}_{0,n} \to \overline{M}_{0,\{i,j,k,l\}} \cong \mathbf{P}^1,$$

and noting that the divisor $D(i, j \mid k, l) \subset \overline{M}_{0,n}(X, \beta)$ is the multiplicity-free inverse image of the point $D(i, j \mid k, l) \in \overline{M}_{0,i,j,k,l}$. The deformation methods of section 5 can be used to prove that the inverse image of the point $D(i, j \mid k, l) \in \overline{M}_{0,i,j,k,l}$ is multiplicity-free. Since points are linearly equivalent on $\mathbf{P}^1$, the linear equivalence on $\overline{M}_{0,n}(X, \beta)$ is established. □

The varieties $\overline{M}_{0,n}(X, \beta)$ come equipped with n morphisms $\rho_1, \ldots, \rho_n$ to X, where ρ_i takes the point $[C, p_1, \ldots, p_n, \mu] \in \overline{M}_{0,n}(X, \beta)$ to the point $\mu(p_i)$ in X. Given arbitrary classes $\gamma_1, \ldots, \gamma_n$ in A^*X, we can construct the cohomology class

$$\rho_1{}^*(\gamma_1) \cup \cdots \cup \rho_n{}^*(\gamma_n)$$

on $\overline{M}_{0,n}(X, \beta)$, and we can evaluate its homogeneous component of the top codimension on the fundamental class, to produce a number, called a *Gromov-Witten invariant*, that we denote by $I_\beta(\gamma_1 \cdots \gamma_n)$:

$$I_\beta(\gamma_1 \cdots \gamma_n) = \int_{\overline{M}_{0,n}(X,\beta)} \rho_1{}^*(\gamma_1) \cup \cdots \cup \rho_n{}^*(\gamma_n). \tag{36}$$

If the classes γ_i are homogeneous, this will be a nonzero number only if the sum of their codimensions is the dimension of $\overline{M}_{0,n}(X,\beta)$, that is,

$$\sum \operatorname{codim}(\gamma_i) = \dim X + \int_\beta c_1(T_X) + n - 3.$$

It follows from the definition that $I_\beta(\gamma_1 \cdots \gamma_n)$ is invariant under permutations of the classes $\gamma_1, \ldots, \gamma_n$. We always assume $n \geq 3$.

The following three special cases are needed:
(I) $\beta = 0$. In this case, $\overline{M}_{0,n}(X,\beta) = \overline{M}_{0,n} \times X$, and the mappings ρ_i are all equal to the projection p onto the second factor. Since

$$\rho_1{}^*(\gamma_1) \cup \cdots \cup \rho_n{}^*(\gamma_n) = p^*(\gamma_1 \cup \cdots \cup \gamma_n),$$

$$\begin{aligned} I_\beta(\gamma_1 \cdots \gamma_n) &= \int_{\overline{M}_{0,n} \times X} p^*(\gamma_1 \cup \cdots \cup \gamma_n) \\ &= \int_{p_*[\overline{M}_{0,n} \times X]} \gamma_1 \cup \cdots \cup \gamma_n . \end{aligned}$$

If $n > 3$, $p_*[\overline{M}_{0,n} \times X] = 0$, since the fibers of p have positive dimension. The only way the number $I_\beta(\gamma_1 \cdots \gamma_n)$ can be nonzero is when $n = 3$, so that $\overline{M}_{0,n}$ is just a point. In this case, $I_\beta(\gamma_1 \cdot \gamma_2 \cdot \gamma_3)$ is the classical intersection number $\int_X \gamma_1 \cup \gamma_2 \cup \gamma_3$.
(II) $\gamma_1 = 1 \in A^0 X$. In this case, $I_\beta(\gamma_1 \cdots \gamma_n)$ vanishes unless $\beta = 0$ and $n = 3$. By (I), $I_0(1 \cdot \gamma_2 \cdot \gamma_3) = \int_X \gamma_2 \cup \gamma_3$. If $n \geq 4$ or $\beta \neq 0$, then the product $\rho_1{}^*(\gamma_1) \cup \cdots \cup \rho_n{}^*(\gamma_n)$ is the pullback of a class on $\overline{M}_{0,n-1}(X,\beta)$ by the map from $\overline{M}_{0,n}(X,\beta)$ to $\overline{M}_{0,n-1}(X,\beta)$ that forgets the first point. Since the fibers of this map have positive dimension, the evaluation of any such class on the fundamental class must vanish.
(III) $\gamma_1 \in A^1 X$, and $n \geq 4$. In this case,

$$I_\beta(\gamma_1 \cdots \gamma_n) = \left(\int_\beta \gamma_1 \right) \cdot I_\beta(\gamma_2 \cdots \gamma_n).$$

The idea for this is that, for a map $\mu : C \to X$ with $\mu_*([C]) = \beta$, there are $\left(\int_\beta \gamma_1\right)$ choices for the point p_1 in C to map to a point in Γ_1, where Γ_1 is a variety representing γ_1. For a formal argument, consider the mapping

$$\psi \ \ \overline{M}_{0,n}(X,\beta) \to X \times \overline{M}_{0,n-1}(X,\beta)$$

which is the product of ρ_1 and the map that forgets the first point. We can write $\psi_*[\overline{M}_{0,n}(X,\beta)] = \beta' \times [\overline{M}_{0,n-1}(X,\beta)] + \alpha$, where β' is a class in $A_1 X$, and α is some class that is supported over a proper algebraic subset of $\overline{M}_{0,n-1}(X,\beta)$. The class β' can be calculated by restricting to what happens over a generic point of $\overline{M}_{0,n-1}(X,\beta)$. Representing this point by $(C, p_2, \ldots, p_n, \mu)$, one sees that $\beta' = \mu_*([C]) = \beta$. Using the projection formula as in (I) and (II), it follows that

$$\begin{aligned} I_\beta(\gamma_1 \cdots \gamma_n) &= \int_{\beta \times [\overline{M}_{0,n-1}(X,\beta)]} \gamma_1 \times \rho_2{}^*(\gamma_2) \cup \cdots \cup \rho_n{}^*(\gamma_n) \\ &= \int_\beta \gamma_1 \cdot \int_{\overline{M}_{0,n-1}(X,\beta)} \rho_2{}^*(\gamma_2) \cup \cdots \cup \rho_n{}^*(\gamma_n), \end{aligned}$$

as asserted.

Let us assume now that X is a homogeneous space. The tangent bundle T_X is generated by its global sections, so $H^1(\mathbf{P}^1, \mu^*(T_X)) = 0$ for all maps μ from $\mathbf{P}^1$ to X. Write $X = G/P$ so G acts transitively on X. The cohomology $A^* X$ has a finite

basis of classes that are represented by subvarieties, and these subvarieties can be put in general position with respect to the action of G.

Let $\Gamma_1, \ldots, \Gamma_n$ be pure dimensional subvarieties of X. Let $[\gamma_i] \in A^*X$ be the corresponding classes (see our notational conventions in section 0.2). Assume

$$\sum_{i=1}^{n} \operatorname{codim}(\Gamma_i) = \dim(X) + \int_\beta c_1(T_X) + n - 3.$$

Let $g\Gamma_i$ denote the g-translate of Γ_i for $g \in G$.

LEMMA 13. *Let $g_1, \ldots, g_n \in G$ be general elements. Then the scheme theoretic intersection*

$$\rho_1^{-1}(g_1\Gamma_1) \cap \cdots \cap \rho_n^{-1}(g_n\Gamma_n) \tag{37}$$

is a finite number of reduced points supported in the automorphism-free locus of $M_{0,n}(X,\beta)$ and

$$I_\beta(\gamma_1 \cdots \gamma_n) = \#\ \rho_1^{-1}(g_1\Gamma_1) \cap \cdots \cap \rho_n^{-1}(g_n\Gamma_n).$$

Proof. Note that $\gamma_i = [g_i\Gamma_i]$ for any $g_i \in G$. Since the moduli space $\overline{M}_{0,n}(X,\beta)$ has finite quotient singularities, there is still a cap product isomorphism

$$A^d(\overline{M}_{0,n}(X,\beta),\mathbb{Q}) \stackrel{\sim}{\to} A_{\dim - d}(\overline{M}_{0,n}(X,\beta),\mathbb{Q})$$

compatible with the cup product (in cohomology) and well-defined intersection product (in homology). We see the classes $\rho_i^*(\gamma_i)$ and $[\rho^{-1}(g_i\Gamma_i)]$ correspond under the above isomorphism. Therefore

$$I_\beta(\gamma_1 \cdots \gamma_n) = \rho_1^{-1}(g_1\Gamma_1) \cap \cdots \cap \rho_n^{-1}(g_n\Gamma_n)$$

where the right side is the intersection product. It will be shown this scheme theoretic intersection is the finite union of reduced points.

Since $n \geq 3$, simple arguments show that the locus of maps with non-trivial automorphisms are of codimension at least 1 in $\overline{M}_{0,n}(X,\beta)$. By transversality arguments (with respect to the G-action on X), it follows the intersection (37) is supported in the automorphism-free locus. Since the automorphism-free locus is nonsingular, the transversality results of Bertini and Kleiman [Kl] show the scheme theoretic intersection is a set of reduced points.

Since the boundary of $\overline{M}_{0,n}(X,\beta)$ is of codimension 1, the intersection (37) is supported in $M_{0,n}(X,\beta)$. □

Lemma 13 relates the Gromov-Witten invariants to enumerative geometry. We see $I_\beta(\gamma_1 \cdots \gamma_n)$ equals the number of pointed maps from $\mathbf{P}^1$ to X representing the class $\beta \in A_1X$ and satisfying $p_i \in g_i\Gamma_i$. If the general element of $M_{0,n}(X,\beta)$ corresponds to a birational map of $\mathbf{P}^1$, then the Gromov-Witten invariant is the number of rational curves in class β passing through general translates of the subvarieties Γ_i.

Let $T_0 = 1 \in A^0X$, let $T_1, \ldots, T_p$ be a basis of A^1X, and let $T_{p+1}, \ldots, T_m$ be a basis for the other cohomology groups. The fundamental numbers counted by these I_β's are the numbers

$$N(n_{p+1}, \ldots, n_m; \beta) = I_\beta({T_{p+1}}^{n_{p+1}} \cdots {T_m}^{n_m}).$$

This is nonzero only when $\sum n_i\,(\operatorname{codim}(\gamma_i) - 1) = \dim X + \int_\beta c_1(T_X) - 3$. In this case, it is the number of pointed rational maps meeting n_i general representatives of γ_i for each i, $p+1 \leq i \leq m$.

An example illustrates the difference between the geometry of maps and the geometry of subcurves. Consider the complete flag variety $\mathbf{Fl}(\mathbb{C}^3)$, which is the space of pairs (p, l) satisfying $p \in l$ where p and l are a point and a line in $\mathbf{P}^2$. Let $\beta \in A_1\mathbf{Fl}(\mathbb{C}^3)$ be the class of the curve $\mathbf{P}^1 \hookrightarrow \mathbf{Fl}(\mathbb{C}^3)$ determined by all pairs (p, l) for a fixed line l. $\overline{M}_{0,0}(\mathbf{Fl}(\mathbb{C}^3), \beta)$ is simply the space of all such embeddings of $\mathbf{P}^1$ determined by lines in $\mathbf{P}^2$. One easily computes:

$$\int_\beta c_1(T_{\mathbf{Fl}(\mathbb{C}^3)}) = 2.$$

The dimension of $\overline{M}_{0,0}(\mathbf{Fl}(\mathbb{C}^3), \beta)$ can be computed by the formula in Theorem 2 ($\dim = 3 + 2 - 3 = 2$) or by observing the space of lines in $\mathbf{P}^2$ is 2 dimensional. $\overline{M}_{0,0}(\mathbf{Fl}(\mathbb{C}^3), \beta)$ has no boundary. Next, consider the space $\overline{M}_{0,0}(\mathbf{Fl}(\mathbb{C}^3), 2\beta)$. The dimension of this moduli space is $3 + 4 - 3 = 4$. Elements of $M_{0,0}(\mathbf{Fl}(\mathbb{C}^3), 2\beta)$ can be obtained by composing a degree 2 map $\mathbf{P}^1 \to \mathbf{P}^1$ with an element of $\overline{M}_{0,0}(\mathbf{Fl}(\mathbb{C}^3), \beta)$. The dimension of the space of these double covers is

$$2 + \dim \overline{M}_{0,0}(\mathbf{Fl}(\mathbb{C}^3), \beta) = 4.$$

It is seen that every element of $M_{0,0}(\mathbf{Fl}(\mathbb{C}^3), 2\beta)$ corresponds to a double cover of an element of $\overline{M}_{0,0}(\mathbf{Fl}(\mathbb{C}^3), \beta)$. The boundary of $\overline{M}_{0,0}(\mathbf{Fl}(\mathbb{C}^3), 2\beta)$ consists of degenerate double covers. Hence, $\overline{M}_{0,0}(\mathbf{Fl}(\mathbb{C}^3), 2\beta)$ parametrizes 2 to 1 maps. Since the space of image curves of maps in $\overline{M}_{0,0}(\mathbf{Fl}(\mathbb{C}^3), 2\beta)$ is only 2 dimensional, it follows that all Gromov-Witten invariants of $\mathbf{Fl}(\mathbb{C}^3)$ of the form $I_{2\beta}(\gamma_1 \cdots \gamma_n)$ vanish. Note also that every element of $\overline{M}_{0,0}(\mathbf{Fl}(\mathbb{C}^3), 2\beta)$ has a nontrivial automorphism.

8. Quantum cohomology

The Gromov-Witten invariants are used to construct a quantum ring structure. It is now assumed that X is a homogeneous space. The precise facts used about X are:

(i) X is a nonsingular, projective, convex variety.
(ii) The Chow and topological homology theories of X are isomorphic.
(iii) The set of effective classes $\beta \in A_1(X)$ (see section 6.1 for the definition) is the cone of non-negative linear combinations of a finite number of effective classes.

For homogeneous spaces, convexity is obtained from the transitive group action. Conditions (ii) and (iii) follow from the Schubert calculus for homogeneous spaces.

We keep the notation of section 7. In addition, define the numbers $g_{ij}, 0 \leq i, j \leq m$, by the equations

$$g_{ij} = \int_X T_i \cup T_j. \tag{38}$$

Define $\left(g^{ij}\right)$ to be the inverse matrix to the matrix (g_{ij}). Equivalently, the class of the diagonal Δ in $X \times X$ is given by the formula

$$[\Delta] = \sum_{e\,f} g^{e\,f}\, T_e \otimes T_f \tag{39}$$

in $A^*(X \times X) = A^*X \otimes A^*X$. The following equations hold:

$$T_i \cup T_j = \sum_{e,\,f} \left(\int_X T_i \cup T_j \cup T_e \right) g^{e\,f} T_f = \sum_{e,\,f} I_0(T_i \cdot T_j \cdot T_e) g^{e\,f} T_f. \tag{40}$$

The idea is to define a "quantum" deformation of the cup multiplication of (40) by allowing nonzero classes β. Here enters a key idea from physics – to write down a "potential function" that carries all the enumerative information.

Define, for a class γ in A^*X,

$$\Phi(\gamma) = \sum_{n\geq 3} \frac{1}{n!} \sum_{\beta} I_\beta(\gamma^n), \tag{41}$$

where γ^n denotes $\gamma \cdots \gamma$ (n times).

LEMMA 14. *For a given integer n, there are only finitely many effective classes $\beta \in A_1 X$ such that $I_\beta(\gamma^n)$ is not zero.*

Proof. Since X is a homogeneous space, the effective classes in A_1X are the non-negative linear combination of finitely many (nonzero) effective classes $\beta_1, \ldots, \beta_p$. By Lemma 11, $\int_{\beta_i} c_1(T_X) \geq 2$. Hence, for a given integer N, there are only a finite number of effective β for which $\int_\beta c_1(T_X) \leq N$. If $I_\beta(\gamma^n)$ is nonzero, then

$$\dim \overline{M}_{0,n}(X,\beta) \leq n \cdot \dim X$$

which implies that $\int_\beta c_1(T_X) \leq (n-1)\cdot \dim X + 3 - n$. □

Let $\gamma = \sum y_i T_i$. By Lemma 14, $\Phi(\gamma) = \Phi(y_0, \ldots, y_m)$ becomes a formal power series in $\mathbb{Q}[[y]] = \mathbb{Q}[[y_0, \ldots, y_m]]$:

$$\Phi(y_0, \ldots, y_m) = \bigoplus_{n_0+\ldots+n_m\geq 3} \sum_{\beta} I_\beta({T_0}^{n_0} \cdots {T_m}^{n_m}) \frac{{y_0}^{n_0}}{n_0!} \cdots \frac{{y_m}^{n_m}}{n_m!}. \tag{42}$$

Define Φ_{ijk} to be the partial derivative:

$$\Phi_{ijk} = \frac{\partial^3 \Phi}{\partial y_i \, \partial y_j \partial y_k}\ , \quad 0 \leq i,j,k \leq m. \tag{43}$$

A simple formal calculation, using (42), gives the following equivalent formula:

$$\Phi_{ijk} = \sum_{n\geq 0} \sum_{\beta} \frac{1}{n!} I_\beta(\gamma^n \cdot T_i \cdot T_j \cdot T_k). \tag{44}$$

Now we define a new "quantum" product $*$ by the rule:

$$T_i * T_j = \sum_{e,f} \Phi_{ije}\, g^{ef}\, T_f. \tag{45}$$

The product in (45) is extended $\mathbb{Q}[[y]]$-linearly to the $\mathbb{Q}[[y]]$-module $A^*X \otimes_{\mathbb{Z}} \mathbb{Q}[[y]]$, thus making it a $\mathbb{Q}[[y]]$-algebra. One thing is evident from this remarkable definition: this product is commutative, since the partial derivatives are symmetric in the subscripts.

It is less obvious, but not difficult, to see $T_0 = 1$ is a unit for the $*$–multiplication. In fact, it follows from property (I) from section 7, together with (44), that

$$\Phi_{0jk} \;=\; I_0(T_0 \cdot T_j \cdot T_k) \;=\; \int_X T_j \cup T_k \;=\; g_{jk},$$

and from this we see that $T_0 * T_j = \sum g_{je}\, g^{ef}\, T_f = T_j$.

The essential point, however, is the associativity:

THEOREM 4. *This definition makes $A^*X \otimes \mathbb{Q}[[y]]$ into a commutative, associative $\mathbb{Q}[[y]]$-algebra, with unit T_0.*

We start the proof by writing down what associativity says:

$$(T_i * T_j) * T_k = \sum_{e,\,f} \Phi_{ije}\, g^{ef}\, T_f * T_k = \sum_{e,\,f} \sum_{c,\,d} \Phi_{ije}\, g^{ef}\, \Phi_{fkc}\, g^{cd}\, T_d,$$

$$T_i * (T_j * T_k) = \sum_{e,\,f} \Phi_{jke}\, g^{ef}\, T_i * T_f = \sum_{e,\,f} \sum_{c,\,d} \Phi_{jke}\, g^{ef}\, \Phi_{ifc}\, g^{cd}\, T_d.$$

Since the matrix $\left(g^{c\,d}\right)$ is nonsingular, the equality of $(T_i * T_j) * T_k$ and $T_i * (T_j * T_k)$ is equivalent to the equation

$$\sum_{e,\,f} \Phi_{ije}\, g^{ef}\, \Phi_{fkl} = \sum_{e,\,f} \Phi_{jke}\, g^{ef}\, \Phi_{ifl}$$

for all l. If we set

$$F(i,j \mid k,l) = \sum_{e,\,f} \Phi_{ije}\, g^{ef}\, \Phi_{fkl}, \tag{46}$$

and use the symmetry $\Phi_{ifl} = \Phi_{fil}$, we see that the associativity is equivalent to the equation

$$F(i,j \mid k,l) = F(j,k \mid i,l). \tag{47}$$

It follows from (44) that

$$F(i,j \mid k,l) = \sum \frac{1}{n_1!\,n_2!} I_{\beta_1}(\gamma^{n_1} \cdot T_i \cdot T_j \cdot T_e)\, g^{ef}\, I_{\beta_2}(\gamma^{n_2} \cdot T_k \cdot T_l \cdot T_f), \tag{48}$$

where the sum is over all nonnegative n_1 and n_2, over all β_1 and β_2 in A_1X, and over all e and f from 0 to m. We need the following lemma. Recall from section 6, the divisor $D(A,B;\beta_1,\beta_2)$. In case A and B are nonempty,

$$D(A,B;\beta_1,\beta_2) = \overline{M}_{0,\,A\cup\{\bullet\}}(X,\beta_1) \times_X \overline{M}_{0,\,B\cup\{\bullet\}}(X,\beta_2).$$

Lemma 15. *Let ι denote the natural inclusion of $D(A,B;\beta_1\beta_2)$ in the Cartesian product $\overline{M}_{0,\,A\cup\{\bullet\}}(X,\beta_1) \times \overline{M}_{0,\,B\cup\{\bullet\}}(X,\beta_2)$, and let α be the embedding of $D(A,B;\beta_1,\beta_2)$ as a divisor in $\overline{M}_{0,n}(X,\beta)$, with $\beta = \beta_1 + \beta_2$. Then for any classes $\gamma_1, \ldots, \gamma_n$ in A^*X,*

$$\iota_* \circ \alpha^*(\rho_1{}^*(\gamma_1) \cup \cdots \cup \rho_n{}^*(\gamma_n)) =$$

$$\sum_{e,\,f} g^{ef} \left(\prod_{a \in A} \rho_a{}^*(\gamma_a) \cdot \rho_\bullet{}^*(T_e) \right) \times \left(\prod_{b \in B} \rho_b{}^*(\gamma_b) \cdot \rho_\bullet{}^*(T_f) \right).$$

Proof. Let $M_1 = \overline{M}_{0,\,A\cup\{\bullet\}}(X,\beta_1)$, $M_2 = M_{0,\,B\cup\{\bullet\}}(X,\beta_2)$, $M = \overline{M}_{0,n}(X,\beta)$, and $D = D(A,B;\beta_1\beta_2)$. From the identification of D with $M_1 \times_X M_2$, we have a commutative diagram, with the right square a fiber square:

$$\begin{array}{ccccc} M & \xleftarrow{\alpha} & D & \xrightarrow{\iota} & M_1 \times M_2 \\ {\scriptstyle\rho}\downarrow & & {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\rho'} \\ X^n & \xleftarrow[p]{} & X^{n+1} & \xrightarrow[\delta]{} & X^{n+2} \end{array} \tag{49}$$

Here ρ is the product of the evaluation maps denoted ρ_i, ρ' is the product of maps ρ_i and the two others denoted $\rho_\bullet$, δ is the diagonal embedding that repeats the last factor, and p is the projection that forgets the last factor. Then we have

$$\begin{aligned}
\iota_* \circ \alpha^*(\rho_1{}^*(\gamma_1) \cup \cdots \cup \rho_n{}^*(\gamma_n)) &= \iota_* \circ \alpha^* \circ \rho^*(\gamma_1 \times \cdots \times \gamma_n) \\
&= \iota_* \circ \eta^* \circ p^*(\gamma_1 \times \ldots \times \gamma_n) \\
&= \iota_* \circ \eta^*(\gamma_1 \times \ldots \times \gamma_n \times [X]) \\
&= \rho'^* \circ \delta_*(\gamma_1 \times \ldots \times \gamma_n \times [X]) \\
&= \rho'^*(\gamma_1 \times \ldots \times \gamma_n \times [\Delta]) \\
&= \sum_{e,\,f} g^{ef} \rho'^*(\gamma_1 \times \ldots \times \gamma_n \times T_e \times T_f)
\end{aligned}$$

$$= \sum_{e,\,f} g^{ef} \left(\prod_{a \in A} \rho_a{}^*(\gamma_a) \cdot \rho_\bullet{}^*(T_e) \right) \times \left(\prod_{b \in B} \rho_b{}^*(\gamma_b) \cdot \rho_\bullet{}^*(T_f) \right).$$

□

Fix $\beta \in A_1 X$ and $\gamma_1, \ldots, \gamma_n \in A^* X$, and fix four distinct integers q, r, s, and t in $[n]$. Set

$$G(q, r \mid s, t) = \sum g^{ef} I_{\beta_1} \left(\prod_{a \in A} \gamma_a \cdot T_e \right) \cdot I_{\beta_2} \left(\prod_{b \in B} \gamma_b \cdot T_f \right), \tag{50}$$

where the sum is over all partitions of $[n]$ into two sets A and B such that q and r are in A and s and t are in B, and over all β_1 and β_2 that sum to β, and over e and f between 0 and m. It follows from Lemma 15 that

$$G(q, r \mid s, t) = \sum \int_{D(A, B; \beta_1, \beta_2)} \rho_1{}^* \gamma_1 \cup \cdots \cup \rho_n{}^* \gamma_n,$$

the sum over A and B and β_1 and β_2 as above. Now Proposition 8 from section 7 implies

$$G(q, r \mid s, t) = G(r, s \mid q, t). \tag{51}$$

Apply (51) in the following case :

$$\gamma_i = \gamma, \quad \text{for } 1 \le i \le n-4,$$
$$\gamma_{n-3} = T_i, \quad \gamma_{n-2} = T_j, \quad \gamma_{n-1} = T_k, \quad \gamma_n = T_l,$$
$$q = n-3, \quad r = n-2, \quad s = n-1, \quad t = n.$$

Then (50) becomes

$$G(q, r \mid s, t) = \sum \binom{n-4}{n_1 - 2} g^{ef} I_{\beta_1}(\gamma^{n_1 - 2} \cdot T_i \cdot T_j \cdot T_e) \cdot I_{\beta_2}(\gamma^{n_2 - 2} \cdot T_k \cdot T_l \cdot T_f),$$

the sum over n_1 and n_2, each at least 2, adding to n, and β_1 and β_2 adding to β; the binomial coefficient is the number of partitions A and B for which A has n_1 elements, and B has n_2 elements. This can be rewritten

$$G(q, r \mid s, t) = n! \sum \frac{1}{n_1!\, n_2!} I_{\beta_1}(\gamma^{n_1} \cdot T_i \cdot T_j \cdot T_e) \cdot I_{\beta_2}(\gamma^{n_2} \cdot T_k \cdot T_l \cdot T_f), \tag{52}$$

the sum over nonnegative n_1 and n_2 adding to $n-4$, and β_1 and β_2 adding to β.

The required equality (47) then follows immediately from (51) and (52), together with (48). This completes the proof of Theorem 4.

9. Applications to enumerative geometry

Using Properties (I)–(III) from the section 7, we can write

$$\Phi(y_0, \ldots, y_m) = \Phi_{\text{classical}}(y) + \Phi_{\text{quantum}}(y),$$

where the classical part is

$$\Phi_{\text{classical}}(y) = \sum_{n_0+\ldots+n_m=3} \int_X (T_0{}^{n_0} \cup \cdots \cup T_m{}^{n_m}) \frac{y_0{}^{n_0}}{n_0!} \cdots \frac{y_m{}^{n_m}}{n_m!}.$$

The quantum part $\Phi_{\text{quantum}}(y)$ is denoted by $\Gamma(y)$:

$$\Gamma(y) = \sum_{n_{p+1}+\ldots+n_m \geq 3} \sum_{\beta \neq 0} N(n_{p+1}, \ldots, n_m; \beta) \prod_{i=1}^{p} e^{\left(\int_\beta T_i\right) y_i} \prod_{i=p+1}^{m} \frac{y_i{}^{n_i}}{n_i!},$$

where $N(n_{p+1}, \ldots, n_m; \beta) = I_\beta(T_{p+1}{}^{n_{p+1}} \cdots T_m{}^{n_m})$. The partial derivatives of $\Phi_{\text{classical}}$ involve only the numbers $\int_X T_i \cup T_j \cup T_k$, while Γ involves the interesting enumerative geometry numbers. From this form of Γ it is easy to read off its partial derivatives.

Let us look again at the projective plane from this point of view. Take the obvious basis: $T_0 = 1$, T_1 the class of a line, and T_2 the class of a point. Note that g_{ij} is 1 if $i + j = 2$, and 0 otherwise, so the same is true for g^{ij}. Therefore

$$T_i * T_j = \Phi_{ij0} T_2 + \Phi_{ij1} T_1 + \Phi_{ij2} T_0.$$

For example,

$$\begin{aligned} T_1 * T_1 &= T_2 + \Gamma_{111} T_1 + \Gamma_{112} T_0, \\ T_1 * T_2 &= \Gamma_{121} T_1 + \Gamma_{122} T_0, \\ T_2 * T_2 &= \Gamma_{221} T_1 + \Gamma_{222} T_0. \end{aligned}$$

Therefore

$$(T_1 * T_1) * T_2 = (\Gamma_{221} T_1 + \Gamma_{222} T_0) + \Gamma_{111}(\Gamma_{121} T_1 + \Gamma_{122} T_0) + \Gamma_{112} T_2,$$

$$T_1 * (T_1 * T_2) = \Gamma_{121}(T_2 + \Gamma_{111} T_1 + \Gamma_{112} T_0) + \Gamma_{122} T_1.$$

The fact that the coefficients of T_0 must be equal in these last two expressions gives the equation

$$\Gamma_{222} = \Gamma_{112}{}^2 - \Gamma_{111}\Gamma_{122}. \tag{53}$$

If $\beta = d[\text{line}]$, the number $N(n, \beta)$ is nonzero only when $n = 3d - 1$, when it is the number N_d of plane rational curves of degree d passing through $3d - 1$ general points. So

$$\Gamma(y) = \sum_{d \geq 1} N_d e^{d y_1} \frac{y_2{}^{3d-1}}{(3d-1)!}.$$

From this we read off the partial derivatives:

$$\begin{aligned}
\Gamma_{222} &= \sum_{d\geq 2} N_d e^{dy_1} \frac{{y_2}^{3d-4}}{(3d-4)!} \\
\Gamma_{112} &= \sum_{d\geq 1} d^2 N_d e^{dy_1} \frac{{y_2}^{3d-2}}{(3d-2)!} \\
\Gamma_{111} &= \sum_{d\geq 1} d^3 N_d e^{dy_1} \frac{{y_2}^{3d-1}}{(3d-1)!} \\
\Gamma_{122} &= \sum_{d\geq 1} d N_d e^{dy_1} \frac{{y_2}^{3d-3}}{(3d-3)!}.
\end{aligned}$$

Therefore

$$\Gamma_{112}{}^2 = \sum_{d\geq 2} \sum_{d_1+d_2=d} {d_1}^2 N_{d_1} {d_2}^2 N_{d_2} e^{dy_1} \frac{{y_2}^{3d-4}}{(3d_1-2)!\,(3d_2-2)!},$$

$$\Gamma_{111}\,\Gamma_{122} = \sum_{d\geq 2} \sum_{d_1+d_2=d} {d_1}^3 N_{d_1} d_2 N_{d_2} e^{dy_1} \frac{{y_2}^{3d-4}}{(3d_1-1)!\,(3d_2-3)!}.$$

In all these sums, d_1 and d_2 are positive. Equating the coefficients of

$$e^{dy_1} {y_2}^{3d-4}/(3d-4)!,$$

we get the identity ($d \geq 2$):

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left[{d_1}^2 {d_2}^2 \binom{3d-4}{3d_1-2} - {d_1}^3 d_2 \binom{3d-4}{3d_1-1} \right]. \tag{54}$$

Here a binomial coefficient $\binom{n}{m}$ is defined to be zero if any of n, m, or $n-m$ is negative. This is the recursion formula discussed in the introduction.

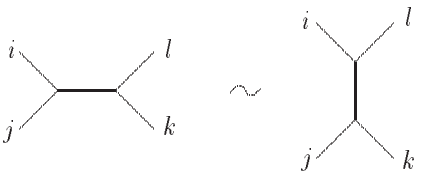
Note that the quantum formalism has removed any necessity to be clever. One simply writes down the associativity equations, and reads off enumerative information. One can organize the information in these associativity equations more systematically as follows (see [DF-I]). Let $F(i,j \mid k,l)$ be defined by (46). For $0 \leq i,j,k,l \leq m$, define:

$$\begin{aligned}
A(i,j,k,l) &= F(i,j \mid k,l) - F(j,k \mid i,l) \\
&= \sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fkl} - \Phi_{jke} g^{ef} \Phi_{fil}.
\end{aligned}$$

Associativity (Theorem 4) amounts to the equations $A(i,j,k,l) = 0$ for all i,j,k,l. The symmetry of Φ_{ijk} in the subscripts and g^{ef} in the superscripts and the basic facts about Φ_{0jk} imply:

(i) $A(k,j,i,l) = -A(i,j,k,l)$,
(ii) $A(l,k,j,i) = A(i,j,k,l)$,
(iii) $A(i,j,k,l) = 0$ if $i = k$ or $j = l$ or if any of the indices i,j,k,l equals 0.

We consider equations equivalent if they differ by sign. For distinct i,j,k,l, the 24 possible equations divide into 3 groups of 8. The equation $A(i,j,k,l) = 0$ that says $F(i,j \mid k,l) = F(j,k \mid i,l)$ can be labelled by a duality diagram from topological field theory (see [DF-I]):

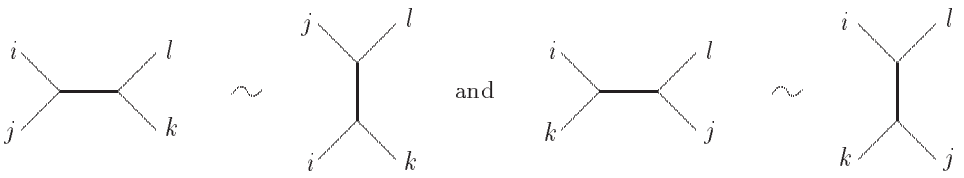


This diagram corresponds to the equations:

$$A(i,j,k,l) = A(j,i,l,k) = A(k,l,i,j) = A(l,k,j,i) = 0$$

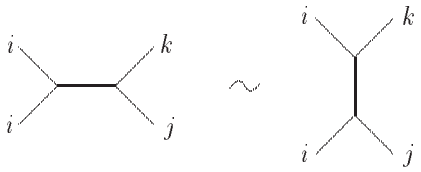
$$-A(i,l,k,j) = -A(k,j,i,l) = -A(l,i,j,k) = -A(j,k,l,i) = 0.$$

To obtain the equations, read the labels around the left or right diagram (either clockwise or counterclockwise, but always reading two grouped together at an end first). The other sixteen equations correspond similarly to the diagrams:

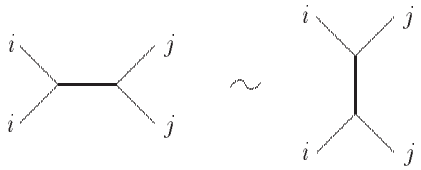


In practice, one only needs to write down one equation for each such diagram.

When 3 of the 4 labels are distinct, say i, i, j, k, there is only 1 equation up to sign (which occurs 8 times). It corresponds to:



When two labels are distinct, there is again only 1 equation up to sign (occurring 4 times):



The symmetry in these diagrams reflects the symmetry in the equations. Taking just one equation for each diagram, one sees that the number $N(m)$ of equations for $\operatorname{rank}(A^*X) = m+1$ is

$$N(m) = 3\binom{m}{4} + m\binom{m-1}{2} + \binom{m}{2} = \frac{m(m-1)(m^2-m+2)}{8},$$

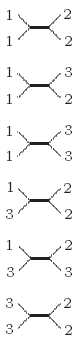
so $N(2) = 1$, $N(3) = 6$, $N(4) = 21$, $N(5) = 55$, $N(6) = 120$, and $N(7) = 231$. For the complete flag manifold $\mathbf{Fl}(\mathbb{C}^n)$, $m = n! - 1$. The number of equations for $\mathbf{Fl}(\mathbb{C}^4)$ is $N(23) = 30861$.

Let us work this out for the two varieties $X = \mathbf{P}^3$ and $X = \mathbf{Q}^3$ (a smooth quadric 3-fold), which have very similar classical cohomology rings. Each has a

basis :

$$\begin{aligned} T_0 &= 1, \\ T_1 &= \text{hyperplane class}, \\ T_2 &= \text{line class}, \\ T_3 &= \text{point class}. \end{aligned}$$

The difference in the classical product is that $T_1 \cup T_1 = T_2$ for $\mathbf{P}^3$ but $T_1 \cup T_1 = 2T_2$ for $\mathbf{Q}^3$. Let $c = 1$ for $\mathbf{P}^3$ and $c = 2$ for $\mathbf{Q}^3$. The $N(3) = 6$ equations are:



$$\begin{aligned} 2\Gamma_{123} - c\Gamma_{222} &= \Gamma_{111}\Gamma_{222} - \Gamma_{112}\Gamma_{122} \\ c\Gamma_{223} + \Gamma_{133} &= \Gamma_{111}\Gamma_{223} - \Gamma_{113}\Gamma_{122} \\ c\Gamma_{223} &= 2\Gamma_{113}\Gamma_{123} - \Gamma_{112}\Gamma_{133} - \Gamma_{111}\Gamma_{233} \\ \Gamma_{223} &= \Gamma_{113}\Gamma_{222} - \Gamma_{112}\Gamma_{223} \\ \Gamma_{333} &= \Gamma_{123}^2 - \Gamma_{122}\Gamma_{133} + \Gamma_{113}\Gamma_{223} - \Gamma_{112}\Gamma_{233} \\ 0 &= \Gamma_{113}\Gamma_{222} - 2\Gamma_{123}\Gamma_{223} + \Gamma_{122}\Gamma_{233} \end{aligned}$$

The function Γ has the form:

$$\Gamma = \sum N_{a,b} e^{dy_1} \frac{y_2^a}{a!} \frac{y_3^b}{b!}. \tag{55}$$

For $\mathbf{P}^3$ the sum in (55) is over non-negative a, b satisfying $a + 2b = 4d$, $d \geq 1$. A crucial difference is that for $\mathbf{Q}^3$, the sum in (55) is over $a + 2b = 3d$, $d \geq 1$ reflecting the fact that $c_1(T_{\mathbf{P}^3}) = 4T_1$ while $c_1(T_{\mathbf{Q}^3}) = 3T_1$. In each case, $N_{a,b}$ is the number of degree d rational curves in X meeting a general lines and b general points of X.

Each of the six differential equations above yields a recursion among the $N_{a,b}$:

(1) For $a \geq 3,\ b \geq 0,\quad 2dN_{a-2,b+1} - cN_{a,b} =$

$$\sum N_{a_1,b_1} N_{a_2,b_2} \binom{b}{b_1} \left(d_1^3 \binom{a-3}{a_1} - d_1^2 d_2 \binom{a-3}{a_1 - 1} \right)$$

(2) For $a \geq 2,\ b \geq 1,\quad dN_{a-2,b+1} - cN_{a,b} =$

$$\sum N_{a_1,b_1} N_{a_2,b_2} \binom{a-2}{a_1} \left(d_1^3 \binom{b-1}{b_1} - d_1^2 d_2 \binom{b-1}{b_1 - 1} \right)$$

(3) For $a \geq 1,\ b \geq 2,\quad cN_{a,b} =$

$$\sum N_{a_1,b_1} N_{a_2,b_2} \left(2d_1^2 d_2 \binom{a-1}{a_1} \binom{b-2}{b_1 - 1} - d_1^2 d_2 \binom{a-1}{a_1 - 1} \binom{b-2}{b_1} \right.$$

$$\left. - d_1^3 \binom{a-1}{a_1} \binom{b-2}{b_2} \right)$$

(4) For $a \geq 3,\ b \geq 1,\quad N_{a-2,b+1} =$

$$\sum N_{a_1,b_1} N_{a_2,b_2} d_1^2 \left(\binom{a-3}{a_1} \binom{b-1}{b_1 - 1} - \binom{a-3}{a_1 - 1} \binom{b-1}{b_1} \right)$$

(5) For $a \geq 2,\ b \geq 2, \quad N_{a-2,b+1} =$

$$\sum N_{a_1,b_1} N_{a_2,b_2} \left(d_1 d_2 \binom{a-2}{a_1-1}\binom{b-2}{b_1-1} - d_1 d_2 \binom{a-2}{a_1-2}\binom{b-2}{b_1} \right.$$
$$\left. + d_1^2 \binom{a-2}{a_1}\binom{b-2}{b_1-1} - d_1^2 \binom{a-2}{a_1-1}\binom{b-2}{b_1} \right)$$

(6) For $a \geq 3,\ b \geq 2, \quad 0 =$

$$\sum N_{a_1,b_1} N_{a_2,b_2} d_1 \left(\binom{a-3}{a_1}\binom{b-2}{b_1-2} - 2\binom{a-3}{a_1-1}\binom{b-2}{b_1-1} \right.$$
$$\left. + \binom{a-3}{a_1-2}\binom{b-2}{b_1} \right)$$

In these formulas, the sum is over non-negative a_1, a_2, b_1, b_2 satisfying

(i) $a_1 + a_2 = a$, $b_1 + b_2 = b$,
(ii) $a + 2b = 4d$, $a_i + 2b_i = 4d_i$, $d_i > 0$ for $\mathbf{P}^3$,
$a + 2b = 3d$, $a_i + 2b_i = 3d_i$, $d_i > 0$ for $\mathbf{Q}^3$.

For $\mathbf{P}^3$, one starts with the $N_{0,2} = 1$ for the number of lines through two points. For $\mathbf{Q}^3$, $N_{1,1} = 1$ is not hard to compute directly. In each case, the six recursions are more than enough to solve for all the other $N_{a,b}$. These numbers for $\mathbf{P}^3$ include the classical results: there are $N_{4,0} = 2$ lines meeting 4 general lines, $N_{8,0} = 92$ conics meeting 8 general lines. and $N_{12,0} = 80160$ twisted cubics meeting 12 general lines. See [DF-I] for more of these numbers[2]. For $\mathbf{Q}^3$, computations yield:

$(d = 1)$ $N_{1,1} = 1$, $N_{3,0} = 1$
$(d = 2)$ $N_{0,3} = 1$, $N_{2,2} = 1$, $N_{4,1} = 2$, $N_{6,0} = 5$
$(d = 3)$ $N_{1,4} = 2$, $N_{3,3} = 5$, $N_{5,2} = 16$, $N_{7,1} = 59$, $N_{9,0} = 242$
$(d = 4)$ $N_{0,6} = 6$, $N_{2,5} = 20$, $N_{4,4} = 74$, $N_{6,3} = 320$, $N_{8,2} = 1546$,
$N_{10,1} = 8148$, $N_{12,0} = 46230$
$(d = 5)$ $N_{1,7} = 106$, $N_{3,6} = 448$, $N_{5,5} = 2180$, $N_{7,4} = 11910$,
$N_{9,3} = 71178$, $N_{11,2} = 457788$, $N_{13,1} = 3136284$,
$N_{15,0} = 22731810$.

The reader is invited to work out the equations for some other simple homogeneous spaces such as $\mathbf{P}^4$, $\mathbf{P}^1 \times \mathbf{P}^1$, $\mathbf{Gr}(2,4)$, or the incidence variety $\mathbf{Fl}(\mathbb{C}^3)$ of points on lines in the plane. For very pleasant excursions along these paths, see [DF-I].

10. Variations

The algebra $A^*X \otimes \mathbb{Q}[[y]]$ may be regarded as the "big" quantum cohomology ring. There is also a "small" quantum cohomology ring that has received some attention, and is often easier to compute explicitly. This amounts to deforming only the classes of divisors, not cohomology classes of higher codimension; and it involves only the 3-point invariants $I_\beta(T_i \cdot T_j \cdot T_k)$. For this, set

$$\overline{\Phi}_{ijk} = \Phi_{ijk}(y_0, y_1, \ldots, y_p, 0, \ldots, 0) = \int_X T_i \cup T_j \cup T_k \ + \ \overline{\Gamma}_{ijk}. \tag{56}$$

[2]The numbers $N_{a,b}$ given in [DF-I] are correct, although their version of equation (6) has a misprint.

The modified quantum potential $\overline{\Gamma}_{ijk}$ is determined by

$$\overline{\Gamma}_{ijk} = \sum_{n\geq 0} \frac{1}{n!} \sum_{\beta\neq 0} I_\beta(\gamma^n \cdot T_i \cdot T_j \cdot T_k)$$

where $\gamma = y_1 T_1 + \ldots + y_p T_p$. By the divisor property (III) of section 7,

$$\overline{\Gamma}_{ijk} = \sum_{\beta\neq 0} I_\beta(T_i \cdot T_j \cdot T_k) q_1^{\int_\beta T_1} \cdots q_p^{\int_\beta T_p}, \tag{57}$$

where $q_i = e^{y_i}$. By Theorem 4, the product

$$T_i * T_j = \sum_{e,\,f} \overline{\Phi}_{ije} g^{ef} T_f \;=\; T_i \cup T_j \;+\; \sum_{e,\,f} \overline{\Gamma}_{ije} g^{ef} T_f$$

then makes the $\mathbb{Z}[q_1, \ldots, q_p]$-module $A^*X \otimes_{\mathbb{Z}} \mathbb{Z}[q_1, \ldots, q_p]$ into a commutative, associative $\mathbb{Z}[q_1, \ldots, q_p]$-algebra, with unit T_0. For example, when $X = \mathbf{P}^r$, T_i is the class of a linear subspace of codimension i, and β is d times the class of a line, the number $I_\beta(T_i \cdot T_j \cdot T_k)$ can be nonzero only if $i + j + k = r + (r+1)d$; this can happen only for $d = 0$ or $d = 1$, and in each case the number is 1. It follows that, with $q = q_1$,

(i) if $i + j \leq r$, then $T_i * T_j = T_{i+j}$.
(ii) if $r + 1 \leq i + j \leq 2r + 1$, $T_i * T_j = q T_{i+j-r-1}$.
(iii) otherwise, $T_i * T_j = 0$.

Therefore the small quantum cohomology ring is

$$\mathbb{Z}[T, q]/(T^{r+1} - q),$$

where $T = T_1$ is the class of a hyperplane.

A similar calculation gives the small quantum cohomology ring of the Grassmannian $X = \mathbf{Gr}(p, n)$ of p-dimensional subspaces of $\mathbb{C}^n$. Let $k = n - p$, let $0 \to S \to \mathbb{C}^n{}_X \to Q \to 0$ be the universal exact sequence of bundles on X, and let $\sigma_i = c_i(Q)$. Set $S_r(\sigma) = \det\left(\sigma_{1+j-i}\right)_{1\leq i,j\leq r}$, and let $q = q_1$.

Proposition 9. *The small quantum cohomology ring of* $\mathbf{Gr}(p, n)$ *is*

$$\mathbb{Z}[\sigma_1, \ldots, \sigma_k, q]/\left(S_{p+1}(\sigma), S_{p+2}(\sigma), \ldots, S_{n-1}(\sigma), S_n(\sigma) + (-1)^k q\right).$$

Proof. Let R denote the small quantum cohomology ring of $\mathbf{Gr}(p, n)$. We use some standard facts about the Grassmannian. In particular, the cohomology has an additive basis of Schubert classes σ_λ, as λ varies over partitions with $k \geq \lambda_1 \geq \ldots \geq \lambda_p \geq 0$; $\sigma_\lambda = [\Omega_\lambda]$ is the class of a Schubert variety

$$\Omega_\lambda = \{L \in X : \dim L \cap V_{k+i-\lambda_i} \geq i \text{ for } 1 \leq i \leq p\},$$

where $V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{C}^n$ is a given flag of subspaces. In $A^*(X)$, $S_r(\sigma)$ represents the r^{th} Chern class of $S^\vee$, from which we have

$$A^*(X) = \mathbb{Z}[\sigma_1, \ldots, \sigma_k]/\left(S_{p+1}(\sigma), \ldots, S_n(\sigma)\right).$$

It therefore suffices to show that the relations displayed in the proposition are valid in R, for then both rings are free over $\mathbb{Z}[q]$ on the same number of generators, and the map from the ring in the proposition to R becomes an isomorphism modulo the ideal generated by q, so it is an isomorphism.

Since $c_1(T_X) = n\sigma_1$, a number $I_\beta(\gamma_1 \cdot \gamma_2 \cdot \gamma_3)$ can be nonzero only if the sum of the codimensions of the γ_i is equal to $\dim X + nd$, where β is d times the class of a line. If $d \geq 1$, all such numbers vanish when $\operatorname{codim}(\gamma_1) + \operatorname{codim}(\gamma_2) < n$. In

particular, the relations $S_i(\sigma) = 0$ for $p < i < n$ remain valid in R. From the formal identity

$$S_n(\sigma) - \sigma_1\, S_{n-1}(\sigma) + \sigma_2\, S_{n-2}(\sigma) - \ldots + (-1)^k \sigma_k\, S_{n-k}(\sigma) = 0,$$

we therefore have $S_n(\sigma) = (-1)^{k-1}\sigma_k\, S_{n-k}(\sigma)$ in R. Since $S_{n-k}(\sigma) = \sigma_{(1^{n-k})}$, the proof will be completed by verifying that $\sigma_k * \sigma_{(1^{n-k})} = q$. Equivalently, when β is the class of a line, we must show that

$$I_\beta(\sigma_k, \sigma_{(1^p)}, \sigma_{(k^p)}) = 1.$$

This is a straightforward calculation. First we have

$$\sigma_k = [\{L : L \supset A\}],\ \sigma_{(1^p)} = [\{L : L \subset B\}],\ \sigma_{(k^p)} = [\{L : L = C\}],$$

where A, B, and C are linear subspaces of $\mathbb{C}^n$ of dimensions 1, $n-1$, and p respectively. It is not hard to verify that any line in X is a Schubert variety of the form $\{L : U \subset L \subset V\}$, where $U \subset V$ are subspaces of $\mathbb{C}^n$ of dimensions $p-1$ and $p+1$. Such a line will meet the three displayed Schubert varieties only if V contains A and C, and U is contained in B and C. For A, B, and C general, there is only one such line, with $U = B \cap C$ and V spanned by A and C. □

This proposition was proved in another way by Bertram [Ber], where the beginnings of some "quantum Schubert calculus" can be found. For the small quantum cohomology ring of a flag manifold, following ideas of Bertram, Givental, and Kim, see [CF][3].

The numbers $I_\beta(\gamma_1 \cdots \gamma_n)$ should not be confused with the numbers denoted by the expression $< \gamma_1, \ldots, \gamma_n >_\beta$ which often occur in discussions of small quantum cohomology rings ([Ber], [CF]). To define the latter, one *fixes* n distinct points $p_1, \ldots, p_n$ in $\mathbf{P}^1$. Then, $< \gamma_1, \ldots, \gamma_n >_\beta$ is the number of maps $\mu : \mathbf{P}^1 \to X$ satisfying: $\mu_*[\mathbf{P}^1] = \beta$ and $\mu(p_i) \in \Gamma_i$ for $1 \le i \le n$ (where Γ_i is a subvariety in general position representing the class γ_i). For $n = 3$, the numbers agree: $I_\beta(\gamma_1 \cdot \gamma_2 \cdot \gamma_3) =< \gamma_1, \gamma_2, \gamma_3 >_\beta$. For $n > 3$, the numbers $< \gamma_1, \ldots, \gamma_n >_\beta$ and $I_\beta(\gamma_1 \cdots \gamma_n)$ are solutions to different enumerative problems. In fact, $< \gamma_1, \ldots, \gamma_n >_\beta$ can be expressed in terms of the 3-points numbers while $I_\beta(\gamma_1 \cdots \gamma_n)$ cannot. For $1 < k < n-1$, $< \gamma_1, \ldots, \gamma_n >_\beta =$

$$\sum_{\beta_1+\beta_2=\beta} \sum_{e,f} < \gamma_1, \ldots, \gamma_k, T_e >_{\beta_1} g^{ef} < T_f, \gamma_{k+1}, \ldots, \gamma_n >_{\beta_2} \tag{58}$$

Equation (58) can be seen geometrically by deforming $\mathbf{P}^1$ to a union of two $\mathbf{P}^1$'s meeting at a point with $p_1, \ldots, p_k$ going to fixed points on the first line and $p_{k+1}, \ldots, p_n$ going to fixed points on the second. Algebraically, in the small quantum cohomology ring,

$$\gamma_1 * \cdots * \gamma_n = \sum_\beta \sum_{e,f} q^\beta < \gamma_1, \ldots, \gamma_n, T_e >_\beta g^{ef} T_f.$$

Equation (58) amounts to the associativity of this product.

We conclude with a few general remarks to relate the discussion and notation here to that in [K-M 1].

[3] A recent preprint of S. Fomin, S. Gelfand, and A. Postnikov, "Quantum Schubert polynomials," extends this Schubert calculus to flag manifolds.

The numbers that we have denoted $I_\beta(\gamma_1 \cdots \gamma_n)$ are part of a more general story. Let η denote the forgetful map from $\overline{M}_{0,n}(X,\beta)$ to $\overline{M}_{0,n}$. For any cohomology classes $\gamma_1, \ldots, \gamma_n$ on X, one can construct a class

$$I^X_{0,\,n,\,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n) = \eta_* \left({\rho_1}^*(\gamma_1) \cup \cdots \cup {\rho_n}^*(\gamma_n)\right) \tag{59}$$

in the cohomology ring $H^*(\overline{M}_{0,n})$. These are called (tree-level, or genus zero) Gromov-Witten classes. The number we denoted $I_\beta(\gamma_1 \cdots \gamma_n)$ is the degree of the zero-dimensional component of this class, which they denote by $\langle I^X_{0,\,n,\,\beta}\rangle(\gamma_1 \otimes \cdots \otimes \gamma_n)$. The intersections with divisors that we have carried out on $\overline{M}_{0,n}(X,\beta)$ can be carried out with the corresponding divisors on $\overline{M}_{0,n}$; this has the advantage that the intersections take place on a nonsingular variety.

One of the main goals of [K-M 1] and especially [K-M 2] is to show how Gromov-Witten classes can be reconstructed from the numbers obtained by evaluating them on the fundamental classes. The idea is that a cohomology class in $H^*(\overline{M}_{0,n})$ is known by evaluating it on the classes of the closures of the strata determined by the combinatorial types of the labeled trees. As we saw and exploited for divisors, these numbers can be expressed in terms of the numbers I_β for the pieces making up the tree.

Kontsevich and Manin also allow cohomology classes of odd degrees, in which case one has to be careful with signs and the ordering of the terms. For an interesting application to some Fano varieties, see [Bea].

Since the space $H = H^*(X,\mathbb{Q})$ can be identified with its dual by Poincaré duality, the maps $I^X_{0,\,n,\,\beta}$ can be regarded as maps

$$H_*(\overline{M}_{0,n+1}) \to \mathrm{Hom}(H^{\otimes n}, H). \tag{60}$$

Both of these, for varying n, have a natural operad structure, that on the first coming from all the ways to glue together labeled trees of $\mathbf{P}^1$'s to form new ones, and the second from all the ways to compose homomorphisms. Remarkably, the associativity (Theorem 4), is equivalent to the assertion that (60) is a morphism of operads.

The structure constants g^{ij} put a metric on the cohomology space $H^*(X,\mathbb{C})$; with coordinates given by the basis for the cohomology, there is a (formal) connection given by the formula $A^k_{ij} = \sum \Phi_{ije} g^{ek}$. In this formalism of Dubrovin, the associativity translates to the assertion that this is a flat connection.

The numbers calculated here are part of a much more ambitious program described in [K-M 1] and [K]. The hope is to extend the story to varieties without the positivity assumptions made here, with some other construction of what should be the fundamental class of $\overline{M}_{g,n}(X,\beta)$. (For varieties whose tangent bundles are not as positive as those considered here, the definition of the potential function Φ is modified by multiplying the summands in (41) by $e^{-\int_\beta \omega}$, for a Kähler class ω, in the hopes of making the power series converge on some open set of the cohomology space H.)

Even if this program is carried out, however – and associativity has been proved by symplectic methods [R-T] in some cases beyond those mentioned here[4] – the interpretation cannot always be in enumerative terms as simple as those we have discussed, cf. [C-M]. On the other hand, these ideas from quantum cohomology

[4] At this conference, J. Li lectured on his work with G. Tian (see [L-T 2]) using cones in vector bundles to construct a "virtual fundamental class" to use in place of $[\overline{M}_{g,n}(X,\beta)]$ in case

have inspired some recent work in enumerative geometry, even in cases where the associativity formalism does not apply directly, cf. [C-H] and [P].

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