THE CALOGERO-SUTHERLAND MODEL AND GENERALIZED CLASSICAL POLYNOMIALS

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Multivariable generalizations of the classical Hermite, Laguerre and Jacobi polynomials occur as the polynomial part of the eigenfunctions of certain Schrödinger operators for Calogero-Sutherland-type quantum systems. For the generalized Hermite and Laguerre polynomials the multidimensional analogues of many classical results regarding generating functions, differentiation and integration formulas, recurrence relations and summation theorems are obtained. We use this and related theory to evaluate the global limit of the ground state density, obtaining in the Hermite case the Wigner semi-circle law, and to give an explicit solution for an initial value problem in the Hermite and Laguerre case.

1 Introduction

The Calogero-Sutherland model refers to exactly solvable quantum many body systems in one-dimension with pair potentials proportional to $1/r^2$ (in some asymptotic limit at least), and which have exact BDJ–type ground states:

$$\psi_0 = \prod_{j=1}^N f_1(x_j) \prod_{1\le j<k\le N} f_2(x_j,x_k). \tag{1.1}$$

Three particular quantum many body systems of the Calogero-Sutherland type are specified by the Schrödinger operators

$$H^{(H)} = -\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \frac{\beta^2}{4}\sum_{j=1}^N x_j^2 + \beta(\beta/2-1)\sum_{1\le j<k\le N}\frac{1}{(x_j-x_k)^2} \tag{1.2a}$$

$$H^{(L)} = -\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^N\left(\frac{\beta a'}{2}\left(\frac{a'\beta}{2}-1\right)\frac{1}{x_j^2} + \frac{\beta^2}{4}x_k^2\right) + 2\beta(\beta/2-1)\sum_{\substack{j,k=1\\ j\ne k}}^N \frac{x_j^2}{(x_k^2-x_j^2)^2} \tag{1.2b}$$

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$$H^{(J)} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial \phi_j^2} + \sum_{j=1}^{N} \left(\frac{a'\beta}{2}\left(\frac{a'\beta}{2} - 1\right) \frac{1}{\sin^2 \phi_j} + \frac{b'\beta}{2}\left(\frac{b'\beta}{2} - 1\right) \frac{1}{\cos^2 \phi_j} \right)$$
$$+2\beta(\beta/2 - 1) \sum_{\substack{j,k=1 \\ j\neq k}}^{N} \frac{\sin^2 \phi_j \cos^2 \phi_k}{(\sin^2 \phi_j - \sin^2 \phi_k)^2}. \tag{1.2c}$$

The superscripts (H), (L), (J) stand for Hermite, Laguerre and Jacobi respectively, and are chosen because of the relationship of these Schrödinger operators to generalizations of the corresponding classical polynomials.

A direct calculation shows that there are eigenfunctions of the form $e^{-\beta W/2}$ where

$$W^{(H)} = \frac{1}{2}\sum_{j=1}^{N} x_j^2 - \sum_{1\le j<k\le N} \log |x_k - x_j| \tag{1.3a}$$
$$W^{(L)} = \frac{1}{2}\sum_{j=1}^{N} x_j^2 - \frac{a'}{2}\sum_{j=1}^{N} \log x_j^2 - \sum_{1\le j<k\le N} \log |x_k^2 - x_j^2| \tag{1.3b}$$
$$W^{(J)} = -\frac{a'}{2}\sum_{j=1}^{N} \log \sin^2 \phi_j - \frac{b'}{2}\sum_{j=1}^{N} \log \cos^2 \phi_j - \sum_{1\le j<k\le N} \log |\sin^2 \phi_j - \sin^2 \phi_k| \tag{1.3c}$$

Since these eigenfunctions are non-negative they correspond to the ground state wavefunction ψ_0 (i.e. they are the eigenfunctions with the most negative eigenvalue E_0). Notice that ψ_0 is indeed of the type (1.1).

Conjugation of the Schrödinger operators by the reciprocal of the ground state $e^{\beta W/2}$ gives the Fokker-Planck operators

$$\mathcal{L} := -\frac{1}{\beta} e^{-\beta W/2}(H - E_0)e^{\beta W/2} = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(\frac{\partial W}{\partial x_j} + \frac{1}{\beta}\frac{\partial}{\partial x_j}\right) \tag{1.4}$$

(for $H^{(J)}$ the coordinates x_j are to replaced by ϕ_j). Thus the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(\{x_j\};t) = H\psi(\{x_j\};t), \tag{1.5}$$

with $\psi = e^{-iE_0 t}e^{\beta W/2}P$ and $t = \tau/i\beta$ transforms to the Fokker-Planck equation

$$\frac{\partial}{\partial \tau}P = \mathcal{L}P. \tag{1.6}$$

The Fokker-Planck equation (1.6) describes the evolution of a classical gas in one-dimension with potential energy W undergoing Brownian motion.

Two classes of problems associated with the Schrödinger operators (1.2) or equivalently the Fokker-Planck operator (1.4) with W given by (1.3), are the topic of this paper. The first is the discussion of some mathematical properties relating to the eigenfunctions, while the second is the evaluation of the density in the ground state and the exact solution of (1.6) for certain initial conditions. These problems are in fact inter-related; we find that the density for each system can be written in terms of a certain eigenstate and that a summation theorem for the eigenstates gives an exact solution of (1.6).

A feature of the Schrödinger operators (1.2) is that after conjugation with the ground state:

$$- e^{\beta W/2}(H - E_0)e^{-\beta W/2} = \sum_{j=1}^{N} \Big(\frac{\partial^2}{\partial x_j^2} - \beta \frac{\partial W}{\partial x_j}\frac{\partial}{\partial x_j} \Big) \tag{1.7}$$

the resulting differential operator has a complete set of polynomial eigenfunctions. In Section 2 we consider the form of the expansion of these polynomials in terms of some different bases of symmetric functions. We note that in the $N = 1$ case, after a suitable change of variables, the operator (1.7) with W given by (1.3) is the eigenoperator for the classical Hermite, Laguerre and Jacobi polynomials. Previous studies of the operator for general N in the Jacobi case [1] have established an orthogonality relation. Since the polynomials in the Hermite and Laguerre cases are limiting cases of these generalized Jacobi polynomials, we can obtain the corresponding orthogonality relations via the limiting procedure.

The generalized Hermite polynomials, which are the polynomial eigenfunctions of (1.4) with $W = W^{(H)}$ as given by (1.3a), are studied in Section 3. Many higher-dimensional analogues of properties of the classical Hermite polynomials are obtained, including a generating function formula, differentiation and integration formulas, a summation theorem and recurrence relations. An analogous study of the generalized Laguerre polynomials is performed in Section 4. In Section 5 we relate the problem of computing the ground state density for the Schrödinger operators (1.2) to the computation of particular eigenstates. By using integral formulas for these eigenstates we are able to compute the global density limit for even values of the coupling β. In the case of the Schrödinger operator (1.2a), the limiting global density is the well known Wigner semi-circle law. Also in Section 5, we give interpretation to results obtained in Sections 3 and 4 for a summation formula. The interpretation is in terms of the solution of an initial value problem associated with the Schrödinger equation (1.5)

We conclude in Section 6 by identifying the formulas contained herein which are to be found in previous works, and give reference to these works (two of the most important references in this regard are unpublished, handwritten manuscripts). In the Appendix we present some results relating to generalized hypergeometric functions depending on two sets of variables which are of relevance to the working in Sections 3 and 4.

2 Inter-relationships

Let us begin by explicitly calculating the operator (1.7) for W given by (1.3). In all cases it is convenient to first change variables: for $W = W^{(H)}$ set $y_j := \sqrt{\beta/2}\, x_j$, for $W = W^{(L)}$ set $y_j = \beta x_j^2/2$, while for $W = W^{(J)}$ set $y_j = \sin^2 \phi_j$. We then obtain

$$\begin{aligned} \tilde{H}^{(H)} &:= -\frac{2}{\beta} e^{\beta W^{(H)}/2}(H^{(H)} - E_0)e^{-\beta W^{(H)}/2} \\ &= \sum_{j=1}^{N} \Big(\frac{\partial^2}{\partial y_j^2} - 2y_j \frac{\partial}{\partial y_j} + \frac{2}{\alpha} \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{1}{y_j - y_k} \frac{\partial}{\partial y_j} \Big), \qquad (2.1a) \\ \tilde{H}^{(L)} &:= -\frac{1}{2\beta} e^{\beta W^{(L)}/2}(H^{(\mathrm{L})} - E_0)e^{-\beta W^{(L)}/2} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} \Big(y_j \frac{\partial^2}{\partial y_j^2} + (a - y_j + 1) \frac{\partial}{\partial y_j} + \frac{2}{\alpha} \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{y_j}{y_j - y_k} \frac{\partial}{\partial y_j} \Big), && (2.1b) \\
\tilde{H}^{(J)} &:= -\frac{1}{4} e^{\beta W^{(J)}/2} (H^{(J)} - E_0) e^{-\beta W^{(J)}/2} \\
&= \sum_{j=1}^{N} \left(y_j(1-y_j) \frac{\partial^2}{\partial y_j^2} + [a + 1 - y_j(a+b+2)] \frac{\partial}{\partial y_j} + \frac{2}{\alpha} \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{y_j(1-y_j)}{y_j - y_k} \frac{\partial}{\partial y_j} \right) && (2.1c)
\end{aligned}
$$

where

$$
a := (\beta a' - 1)/2, \qquad b := (\beta b' - 1)/2, \qquad \alpha := 2/\beta.
$$

In the one-variable case ($N = 1$), these operators have a complete set of polynomial eigenfunctions given by the classical Hermite, Laguerre and Jacobi polynomials

$$
\begin{aligned}
H_n(y) &:= n! \sum_{j=0}^{[n/2]} \frac{(-1)^j (2y)^{n-2j}}{j!(n-2j)!} && (2.2a) \\
L_n^a(y) &:= \frac{(a+1)_n}{n!} \sum_{j=0}^{n} \binom{n}{j} \frac{(-y)^j}{(a+1)_j} && (2.2b) \\
P_n^{(b,a)}(2y-1) &:= (-1)^n \binom{n+a}{n} \sum_{j=0}^{n} \frac{(n+a+b+1)_j}{(a+1)_j} \binom{n}{j} (-y)^j && (2.2c)
\end{aligned}
$$

respectively, where $(u)_n := u(u+1)\dots(u+n-1)$.

It is also true for general N that there is a complete set of polynomial eigenfunctions for each of the operators (2.1). This can be seen by computing their action on the monomial symmetric polynomial m_κ, where κ denotes a partition consisting of N parts κ_j. We obtain series of the form

$$
\begin{aligned}
e^{(H)}(\kappa, \alpha) m_\kappa &+ \sum_{|\mu| < |\kappa|} b_{\mu\kappa}^{(H)} m_\mu && (2.3a) \\
e^{(L)}(\kappa, \alpha) m_\kappa &+ \sum_{|\mu| < |\kappa|} b_{\mu\kappa}^{(L)} m_\mu && (2.3b) \\
e^{(J)}(\kappa, \alpha) m_\kappa &+ \sum_{\mu < \kappa} a_{\mu\kappa}^{(J)} m_\mu + \sum_{|\mu| < |\kappa|} b_{\mu\kappa}^{(J)} m_\mu && (2.3c)
\end{aligned}
$$

respectively, where the notation $|\mu| < |\kappa|$ means $\sum_{j=1}^N \mu_j < \sum_{j=1}^N \kappa_j$, while the notation $\mu < \kappa$ means $\mu \neq \kappa$ but

$$
\sum_{j=1}^{N} \mu_j = \sum_{j=1}^{N} \kappa_j \quad \text{and} \quad \sum_{j=1}^{p} \mu_j \leq \sum_{j=1}^{p} \kappa_j \quad \text{for each} \quad p = 1, \dots, N
$$

Also, $a_{\mu\kappa}$, $b_{\mu\kappa}$ are coefficients independent of y_j and

$$
e^{(H)}(\kappa, \alpha) = -2|\kappa|, \qquad e^{(L)}(\kappa, \alpha) = -|\kappa| \tag{2.4}
$$

(the explicit value of $e^{(J)}(\kappa, \alpha)$ can also be computed, however it is not needed in our subsequent discussion). This means there are eigenfunctions of the form

$$
\tilde{a}_{\kappa\kappa}^{(H)} m_\kappa + \sum_{|\mu| < |\kappa|} \tilde{b}_{\mu\kappa}^{(H)} m_\mu \tag{2.5a}
$$

$$\tilde{a}^{(L)}_{\kappa\kappa} m_\kappa \quad + \sum_{|\mu|<|\kappa|} \tilde{b}^{(L)}_{\mu\kappa} m_\mu \tag{2.5b}$$

$$\tilde{a}^{(J)}_{\kappa\kappa} m_\kappa \quad + \sum_{\mu<\kappa} \tilde{a}^{(J)}_{\mu\kappa} m_\mu + \sum_{|\mu|<|\kappa|} \tilde{b}^{(J)}_{\mu\kappa} m_\mu \tag{2.5c}$$

with eigenvalues $e^{(H)}(\kappa,\alpha)$, $e^{(L)}(\kappa,\alpha)$ and $e^{(J)}(\kappa,\alpha)$ respectively.

Rather than study the eigenfunctions in the form (2.5), previous studies [24, 19] have shown that it is advantageous to change basis from the monomial symmetric polynomials to the Jack polynomials [30, 24]. We recall that the Jack polynomial $J^{(\alpha)}_\kappa(z_1,\dots,z_N)$ is the unique (up to normalization) symmetric eigenfunction of the operator

$$D_2 := \sum_{j=1}^N z_j^2 \frac{\partial^2}{\partial z_j^2} + \frac{2}{\alpha} \sum_{\substack{j,k=1 \\ j\neq k}}^N \frac{z_j^2}{z_j - z_k} \frac{\partial}{\partial z_j} \tag{2.6}$$

which has an expansion of the form

$$a_{\kappa\kappa} m_\kappa + \sum_{\mu<\kappa} a_{\mu\kappa} m_\mu. \tag{2.7}$$

The notation $J^{(\alpha)}_\kappa$ is usually used for the particular normalization $a_{(1^{|\kappa|})\kappa} = |\kappa|!$ in (2.7). However, for our purposes it is more convenient to choose a different normalization, and to denote the corresponding Jack polynomial by $C^{(\alpha)}_\kappa$ as in e.g. [17]. This normalization is specified by requiring

$$(x_1 + \ldots + x_N)^n = \sum_{|\kappa|=n} C^{(\alpha)}_\kappa(x_1,\dots,x_N) \tag{2.8}$$

It is known (see e.g. [17]) that $J^{(\alpha)}_\kappa$ and $C^{(\alpha)}_\kappa$ are related by

$$C^{(\alpha)}_\kappa(x_1,\dots,x_N) = \alpha^{|\kappa|}|\kappa|! j_\kappa^{-1} J^{(\alpha)}_\kappa(x_1,\dots,x_N) \tag{2.9}$$

where

$$j_\kappa := \prod_{s\in\kappa} h^*_\kappa(s)\, h^\kappa_*(s) \tag{2.10a}$$

with $$h^*_\kappa(s) := l_\kappa(s) + \alpha(a_\kappa(s)+1) \quad h^\kappa_*(s) := l_\kappa(s) + 1 + \alpha a_\kappa(s) \tag{2.10b}$$

In (2.10a) and (2.10b), κ is regarded as a diagram, s denotes a node in the diagram and $a_\kappa(s)$ ($l_\kappa(s)$) denotes the arm length (leg length) of the node (see e.g. [25]). In terms of the Jack polynomials, it is known [24, 21, 23, 22] that for each partition κ there is an eigenfunction of the form

$$H_\kappa(y_1,\dots,y_N;\alpha) := \sum_{\mu\subseteq\kappa} c^{(H)}_{\mu\kappa} C^{(\alpha)}_\mu(y_1,\dots,y_N) \tag{2.11a}$$

$$L^a_\kappa(y_1,\dots,y_N;\alpha) := \sum_{\mu\subseteq\kappa} c^{(L)}_{\mu\kappa} C^{(\alpha)}_\mu(y_1,\dots,y_N) \tag{2.11b}$$

$$G^{(a,b)}_\kappa(y_1,\dots,y_N;\alpha) := \sum_{\mu\subseteq\kappa} c^{(J)}_{\mu\kappa} C^{(\alpha)}_\mu(y_1,\dots,y_N) \tag{2.11c}$$

where the notation $\mu \subseteq \kappa$ denotes $\mu_j \le \kappa_j$ for each $j = 1, \dots, N$ and $c_{\kappa\kappa} \neq 0$. These results can be established by using known formulas for the action of the operators

$$E_k := \sum_{i=1}^N x_i^k \frac{\partial}{\partial x_i} \tag{2.12a}$$

$$D_k := \sum_{i=1}^N x_i^k \frac{\partial^2}{\partial x_i^2} + \frac{2}{\alpha} \sum_{i \neq j} \frac{x_i^k}{x_i - x_j} \frac{\partial}{\partial x_i} \tag{2.12b}$$

for $k = 0, 1, 2$ which for future reference we list here:

$$E_0 \frac{C_\kappa^{(\alpha)}(x)}{C_\kappa^{(\alpha)}(1^N)} = \sum_{i=1}^N \binom{\kappa}{\kappa_{(i)}} \frac{C_{\kappa_{(i)}}^{(\alpha)}(x)}{C_{\kappa_{(i)}}^{(\alpha)}(1^N)} \tag{2.13a}$$

$$E_1 C_\kappa^{(\alpha)}(x) = |\kappa| C_\kappa^{(\alpha)}(x) \tag{2.13b}$$

$$E_2 C_\kappa^{(\alpha)}(x) = \frac{1}{1+|\kappa|} \sum_{i=1}^N \binom{\kappa^{(i)}}{\kappa} \left(\kappa_i - \frac{i-1}{\alpha}\right) C_{\kappa^{(i)}}^{(\alpha)}(x) \tag{2.13c}$$

$$D_1 \frac{C_\kappa^{(\alpha)}(x)}{C_\kappa^{(\alpha)}(1^N)} = \sum_{i=1}^N \binom{\kappa}{\kappa_{(i)}} \left(\kappa_i - 1 + \frac{N-i}{\alpha}\right) \frac{C_{\kappa_{(i)}}^{(\alpha)}(x)}{C_{\kappa_{(i)}}^{(\alpha)}(1^N)} \tag{2.13d}$$

$$D_2 C_\kappa^{(\alpha)}(x) = d_\kappa C_\kappa^{(\alpha)}(x), \quad d_\kappa := \frac{2}{\alpha}|\kappa|(N-1) + \sum_{i=1}^N \kappa_i \left(\kappa_i - 1 - \frac{2}{\alpha}(i-1)\right) \tag{2.13e}$$

(the action of D_0 can be computed from the commutator formula $D_0 = [E_0, D_1]$). Here the generalized binomial coefficients $\binom{\kappa}{\sigma}$ are defined by the expansion

$$\frac{C_\kappa^{(\alpha)}(1+t_1, \dots, 1+t_N)}{C_\kappa^{(\alpha)}(1^N)} = \sum_{s=0}^{|\kappa|} \sum_{|\sigma|=s} \binom{\kappa}{\sigma} \frac{C_\sigma^{(\alpha)}(t_1, \dots, t_N)}{C_\sigma^{(\alpha)}(1^N)} \tag{2.14}$$

where $C_\kappa^{(\alpha)}(1^N)$ has the explicit form

$$C_\kappa^{(\alpha)}(1^N) = \frac{\alpha^{|\kappa|}\, |\kappa|!}{j_\kappa} \prod_{(i,j) \in \kappa} \left(N - (i-1) + \alpha(j-1)\right). \tag{2.15}$$

We have also used the notation

$$\kappa_{(i)} := (\kappa_1, \dots, \kappa_{i-1}, \kappa_i - 1, \kappa_{i+1}, \dots, \kappa_N), \quad \kappa^{(i)} := (\kappa_1, \dots, \kappa_{i-1}, \kappa_i + 1, \kappa_{i+1}, \dots, \kappa_N)$$

(note that this is the opposite of what is used in [27, 17] but rather is that used by [20]).

The polynomials in (2.11) are referred to as generalized Hermite, Laguerre and Jacobi polynomials respectively [14]; they are uniquely specified up to normalization as the eigenfunctions of the operators (2.1) with an expansion in terms of Jack polynomials with highest weight (i.e. largest partition in reverse lexicographical ordering) $c_{\kappa\kappa} C_\kappa^{(\alpha)}$. For the normalization we choose

$$c_{\kappa\kappa}^{(H)} = 2^{|\kappa|} / C_\kappa^{(\alpha)}(1^N), \quad c_{\kappa\kappa}^{(L)} = (-1)^{|\kappa|} / |\kappa|! C_\kappa^{(\alpha)}(1^N) \quad \text{and} \quad c_{\kappa\kappa}^{(J)} = 1. \tag{2.16}$$

With this choice, for $N = 1$ the generalized Hermite and Laguerre polynomials exactly coincide with the classical Hermite and Laguerre polynomials (2.2a) and (2.2b) respectively, while in the $N = 1$ case $G_{(k)}^{(a,b)}$ corresponds to the Jacobi polynomial $P_k^{(a,b)}(2y-1)$, normalized so that the coefficient of y^k is unity.

There have been a number of studies of the generalized Jacobi polynomials $G_\kappa^{(a,b)}$ [22, 4, 24]. In particular, it is known that these polynomials are orthogonal with respect to the inner product

$$\langle f|g\rangle^{(\mathrm{J})} := \prod_{l=1}^N \int_0^1 dy_l\, y_l^a (1-y_l)^b \prod_{1\le j<k\le N} |y_k - y_j|^{2/\alpha} f(y_1,\dots,y_N) g(y_1,\dots,y_N) \tag{2.17}$$

This is significant to the study of the generalized Hermite and Laguerre polynomials, as both are limiting cases of the Jacobi polynomials. Thus by comparing the operators $\tilde{H}^{(H)}$ and $\tilde{H}^{(L)}$ with $\tilde{H}^{(J)}$, and using the facts that the Jack polynomial $C_\kappa^{(\alpha)}$ is homogeneous of order $|\kappa|$ and that in the expansion (2.14) the binomial coefficient is non-zero if and only if $\mu \subset \kappa$ [17], we see that

$$\lim_{b\to\infty} \frac{2^{2|\kappa|}(-b)^{|\kappa|}}{C_\kappa^{(\alpha)}(1^N)} G_\kappa^{(b,b)}\Big(\frac{1}{2}(1-\frac{y_1}{b}),\dots,\frac{1}{2}(1-\frac{y_N}{b})\Big) = H_\kappa(y_1,\dots,y_N;\alpha) \tag{2.18}$$

and

$$\lim_{b\to\infty} \frac{(-1)^{|\kappa|} b^{|\kappa|}}{|\kappa|! C_\kappa^{(\alpha)}(1^N)} G_\kappa^{(a,b)}(y_1/b,\dots,y_N/b;\alpha) = L_\kappa^a(y_1,\dots,y_N;\alpha) \tag{2.19}$$

It thus follows that by performing the same change of variables and limiting procedure in (2.17), we will obtain inner products for which the Hermite and Laguerre polynomials are orthogonal with respect to. We find that for the generalized Hermite polynomials this inner product is

$$\langle f|g\rangle^{(\mathrm{H})} := \prod_{l=1}^N \int_{-\infty}^\infty dy_l\, e^{-y_l^2} \prod_{1\le j<k\le N} |y_k - y_j|^{2/\alpha} f(y_1,\dots,y_N) g(y_1,\dots,y_N) \tag{2.20}$$

while for the generalized Laguerre polynomials it is

$$\langle f|g\rangle^{(\mathrm{L})} := \prod_{l=1}^N \int_0^\infty dy_l\, y_l^a e^{-y_l} \prod_{1\le j<k\le N} |y_k - y_j|^{2/\alpha} f(y_1,\dots,y_N) g(y_1,\dots,y_N). \tag{2.21}$$

(these inner products have previously been identified by Lassalle [21, 23]).

3 The generalized Hermite polynomials

3.1 The generating function

The starting point and key source of inspiration in our studies of the generalized Hermite and Laguerre polynomials is a private correspondence with M. Lassalle[19], in which we received unpublished notes containing, amongst other results, a multi-variable generalization of the classical generating function formula

$$\sum_{k=0}^\infty \frac{H_k(y) z^k}{k!} = e^{2yz} e^{-z^2}, \tag{3.1}$$

which is given by the following result.

Proposition 3.1 *Let $y := (y_1, \dots, y_N)$ and $z := (z_1, \dots, z_N)$. The generalized Hermite polynomials $H_\kappa(y;\alpha)$, defined in the previous section as polynomial eigenfunctions of the operator (2.1a), which have highest weight term as in (2.11a) with the normalization specified by (2.16), are given by the generating function*

$$\sum_\kappa \frac{1}{|\kappa|!} H_\kappa(y;\alpha) C_\kappa^{(\alpha)}(z) = {}_0\mathcal{F}_0^{(\alpha)}(2y;z) e^{-p_2(z)} \tag{3.2a}$$

where

$${}_0\mathcal{F}_0^{(\alpha)}(2y;z) := \sum_\kappa \frac{1}{|\kappa|!} \frac{C_\kappa^{(\alpha)}(2y) C_\kappa^{(\alpha)}(z)}{C_\kappa^{(\alpha)}(1^N)} \quad \text{and} \quad p_2(z) := \sum_{j=1}^N z_j^2. \tag{3.2b}$$

A fundamental result in Lassalle's researches is an explicit formula for the action of the operator $E_0^{(y)}$ (recall (2.13a); the superscript (y) indicates operation with respect to the variables y) on ${}_0\mathcal{F}_0^{(\alpha)}$:

$$E_0^{(y)}\, {}_0\mathcal{F}_0^{(\alpha)}(2y;z) = 2p_1(z)\, {}_0\mathcal{F}_0^{(\alpha)}(2y;z), \quad \text{where} \quad p_1(z) := \sum_{j=1}^N z_j. \tag{3.3}$$

This formula follows from (2.13a) and the result [17, 20]

$$p_1(x)\, C_\kappa^{(\alpha)}(x) = \frac{1}{1+|\kappa|} \sum_{i=1}^N \binom{\kappa^{(i)}}{\kappa} C_{\kappa^{(i)}}^{(\alpha)}(x) \tag{3.4}$$

Now in the notation of (2.12), the operator $\tilde{H}^{(H)}$ (2.1a) is given by

$$\tilde{H}^{(H)} = D_0 - 2E_1 \tag{3.5}$$

Knowledge of the action of $D_0^{(y)}$ on ${}_0\mathcal{F}_0^{(\alpha)}(2y;z)$ is required to prove Proposition 3.1. Lassalle uses the formulas (2.13) and (3.3) to establish this action. We have observed that in fact the required formula can be derived from (3.3). In our derivation we make use of the general fact that if $A^{(y)}F = \hat{A}^{(z)}F$ and $B^{(y)}F = \hat{B}^{(z)}F$, then

$$A^{(y)}B^{(y)}F = A^{(y)}\hat{B}^{(z)}F = \hat{B}^{(z)}A^{(y)}F = \hat{B}^{(z)}\hat{A}^{(z)}F \tag{3.6}$$

where the second equality follows because operators acting on different sets of variables always commute.

Lemma 3.1 *We have*

$$D_1^{(y)}\, {}_0\mathcal{F}_0^{(\alpha)}(2y;z) = \Big(\frac{2}{\alpha}(N-1)p_1(z) + 2E_2^{(z)}\Big){}_0\mathcal{F}_0^{(\alpha)}(2y;z) \tag{3.7a}$$

$$D_0^{(y)}\, {}_0\mathcal{F}_0^{(\alpha)}(2y;z) = 4p_2(z)\,{}_0F_0^{(\alpha)}(2y;z) \tag{3.7b}$$

Proof Since $D_1^{(y)} = \frac{1}{2}[E_0^{(y)}, D_2^{(y)}]$, using (3.3), the fact that $D_2^{(y)}$ is an eigenoperator for the Jack polynomials, and (3.6) gives

$$\begin{aligned} D_1^{(y)}\, {}_0\mathcal{F}_0^{(\alpha)}(2y;z) &= [D_2^{(z)}, p_1(z)]\, {}_0\mathcal{F}_0^{(\alpha)}(2y;z) \\ &= \Big(\frac{2}{\alpha}(N-1)p_1(z) + 2E_2^{(z)}\Big){}_0\mathcal{F}_0^{(\alpha)}(2y;z) \end{aligned}$$

where the second equality follows by computing the commutator. To derive the second result, note that $D_0^{(y)} = [E_0^{(y)}, D_1^{(y)}]$, so from (3.3), (3.7a) and (3.6)

$$\begin{aligned} D_0^{(y)} \, {}_0\mathcal{F}_0^{(\alpha)}(2y;z) &= [\frac{2}{\alpha}(N-1)p_1(z) + 2E_2^{(z)}, 2p_1(z)] \, {}_0\mathcal{F}_0^{(\alpha)}(2y;z) \\ &= 4p_2(z) \, {}_0\mathcal{F}_0^{(\alpha)}(2y;z). \end{aligned}$$

Let us now show how (3.7b) is used in Lassalle's derivation of (3.2a).

Proof of Proposition 3.1 We first want to show that $H_\kappa(y;\alpha)$ as defined by the generating function (3.2a) is an eigenfunction of (3.5) with eigenvalue $-2|\kappa|$. To do this, consider the action of $E_1^{(z)}$ on both sides of (3.2a). On the r.h.s. we have

$$\begin{aligned} & E_1^{(z)} \, {}_0\mathcal{F}_0^{(\alpha)}(2y;z) e^{-p_2(z)} \\ &= e^{-p_2(z)} E_1^{(z)} \, {}_0\mathcal{F}_0^{(\alpha)}(2y;z) - 2p_2(z) {}_0\mathcal{F}_0^{(\alpha)}(2y;z) e^{-p_2(z)} \\ &= e^{-p_2(z)} E_1^{(y)} \, {}_0\mathcal{F}_0^{(\alpha)}(2y;z) - \frac{1}{2} D_0^{(y)} \, {}_0\mathcal{F}_0^{(\alpha)}(2y;z) e^{-p_2(z)} \\ &= -\frac{1}{2} \sum_\kappa \frac{1}{|\kappa|!} (D_0^{(y)} - 2E_1^{(y)}) H_\kappa(y;\alpha) C_\kappa^{(\alpha)}(z), \end{aligned} \tag{3.8}$$

where the second equality follows by using (3.7b) and noting that since E_1 is an eigenoperator of the Jack polynomials, the definition (3.2b) gives

$$E_1^{(y)} \, {}_0F_0^{(\alpha)}(2y;z) = E_1^{(z)} \, {}_0F_0^{(\alpha)}(2y;z),$$

and the final equality follows by substituting the generating function. On the l.h.s., since $E_1^{(z)}$ is an eigenoperator of $C_\kappa^{(\alpha)}(z)$ with eigenvalue $|\kappa|$, from the definition (3.2b)

$$E_1^{(z)} \sum_\kappa \frac{1}{|\kappa|!} H_\kappa(y;\alpha) C_\kappa^{(\alpha)}(z) = \sum_\kappa \frac{1}{|\kappa|!} H_\kappa(y;\alpha) |\kappa| C_\kappa^{(\alpha)}(z) \tag{3.9}$$

Equating coefficients of $C_\kappa^{(\alpha)}(z)$ in (3.8) and (3.9) shows that $H_\kappa(y;\alpha)$ is an eigenfunction of the operator (3.5) with eigenvalue $-2|\kappa|$ as required.

It remains to check that $H_\kappa(y;\alpha)$ as given by (3.2a) has an expansion in terms of Jack polynomials with highest weight term $2^{|\kappa|} C_\kappa^{(\alpha)}(y)/C_\kappa^{(\alpha)}(1^N)$. This follows from the fact that to compute the coefficient of $C_\kappa^{(\alpha)}(z)$ in ${}_0\mathcal{F}_0^{(\alpha)}(2y;z)e^{-p_2(z)}$, the sum in (3.2b) can be restricted to partitions with modulus less than or equal to $|\kappa|$.

The strategy of the above proof leads us to generalizations of the eigenvalue equation. In the theory of Jack polynomials, Macdonald [25] (see also [29, 4]) has given a family of differential operators $\{D_N^j\}_{j=1,\dots,N}$ which have the Jack polynomials as eigenfunctions, and for which the corresponding eigenvalues are known explicitly. These operators are given by

$$\begin{aligned} D_N^p \;:=\; & \sum_{l=0}^{p} (\alpha)^{p-l} \sum_{1 \le i_1 < i_2 < \dots < i_l} \frac{1}{\Delta_+} \Big(z_{i_1} \frac{\partial}{\partial z_{i_1}} \dots z_{i_l} \frac{\partial}{\partial z_{i_l}} \Big) \Delta_+ \\ & \times \sum_{\substack{1 \le i_{l+1} < \dots < i_p \le N \\ \ne i_1 \dots \ne i_l}} \Big(z_{i_{l+1}} \frac{\partial}{\partial z_{i_{l+1}}} \dots z_{i_p} \frac{\partial}{\partial z_{i_p}} \Big), \end{aligned}$$

where $\Delta_+ := \prod_{1\le j<k\le N}(z_k - z_j)$. Furthermore, if we define the corresponding generating function by

$$D_N(X;\alpha) := \sum_{k=0}^{N} X^{N-k} D_N^k$$

then the eigenvalues are given by

$$e(\kappa,\alpha;X) := \prod_{j=1}^{N}(X + N - j + \alpha\kappa_j). \tag{3.10}$$

The operator E_1 is related to the Macdonald operator D_N^1 by $D_N^1 = \alpha E_1 + N(N-1)/2$. By considering the analogue of (3.8) with $E_1^{(z)}$ replaced by D_N^j ($j = 1, 2, \ldots, N$), a family of differential operators which have the generalized Hermite polynomials as eigenfunctions can be given The r.h.s. of (3.8) is then computed according to the Baker-Campbell-Hausdorff formula

$$\begin{aligned} D_N^{j\,(z)}\, f\, e^{p_1(z)} = e^{p_1(z)} \Big(D_N^{j\,(z)} + \left[D_N^{p(z)}, p_1(z)\right] \\ + \frac{1}{2!}\left[\left[D_N^{j\,(z)}, p_1(z)\right], p_1(z)\right] + \cdots \frac{1}{n!}\left[\cdots\left[D_N^{j\,(z)}, p_1(z)\right], \cdots, p_1(z)\right]\Big) f \end{aligned} \tag{3.11}$$

Note that the sum on the r.h.s. terminates after the n-th nested commutator since the highest derivative in $D_N^{j(z)}$ has degree j.

Following the derivation of the eigenvalue equation given in in the proof of Proposition 3.1, and thus using (3.3), (3.6) and the fact that since D_N^j is an eigenoperator of the Jacks we have

$$D_N^{j\,(z)}\,{}_0\mathcal{F}_0^{(\alpha)}(2y;z) = D_N^{j\,(y)}\,{}_0\mathcal{F}_0^{(\alpha)}(2y;z)$$

we can immediately deduce a family of N independent eigenoperators of the polynomials $H_\kappa(y;\alpha)$, together with the corresponding eigenvalues.

Proposition 3.2 *Let*

$$\begin{aligned} \tilde{H}_j^{(H)} &:= D_N^{j(y)} - \frac{1}{4}\left[D_0^{(y)}, D_N^{j(y)}\right] + \frac{1}{4^2\,2!}\left[D_0^{(y)}, \left[D_0^{(y)}, D_N^{j(y)}\right]\right] - \cdots \\ &\quad + \frac{(-1)^n}{4^n\,n!}\left[D_0^{(y)}, \left[D_0^{(y)}, \cdots, \left[D_0^{(y)}, D_N^{j(y)}\right]\right]\cdots\right] \end{aligned}$$

We have that $H_\kappa(y;\alpha)$ is an eigenfunction of $\tilde{H}_j^{(H)}$ for each $j = 1, \ldots, N$, with eigenvalue $e_j(\kappa;\alpha)$ given by the coefficient of X^{N-j} in (3.10).

3.2 Consequences of the generating function

The generating function formula (3.2a) can be used to deduce higher-dimensional analogues of the classical properties of the Hermite polynomials

$$H_n(-y) = (-1)^n H_n(y) \tag{3.12a}$$

$$\frac{d}{dy} H_n(y) = 2n H_{n-1}(y) \tag{3.12b}$$

$$2y H_n(y) = H_{n+1}(y) + 2n H_{n-1}(y). \tag{3.12c}$$

Proposition 3.3 *We have*

$$H_\kappa(-y;\alpha) = (-1)^{|\kappa|}\, H_\kappa(y;\alpha).$$

Proof Replacing y by $-y$ in (3.2a) and using the fact that ${}_0\mathcal{F}_0^{(\alpha)}(2\mu y;z) = {}_0\mathcal{F}_0^{(\alpha)}(2y;\mu z)$, where μ is any scalar, and that $C_\kappa^{(\alpha)}$ is homogeneous of order $|\kappa|$, gives

$$\sum_\kappa \frac{1}{|\kappa|!} H_\kappa(-y;\alpha) C_\kappa^{(\alpha)}(z) = {}_0\mathcal{F}_0^{(\alpha)}(-2y;z)e^{-p_2(z)} = {}_0\mathcal{F}_0^{(\alpha)}(2y;-z)e^{-p_2(-z)}$$

$$= \sum_\kappa \frac{1}{|\kappa|!}(-1)^{|\kappa|} H_\kappa(y;\alpha) C_\kappa^{(\alpha)}(z).$$

The result follows by equating coefficients of $C_\kappa^{(\alpha)}(z)$.

Proposition 3.4 *We have*

$$E_0^{(y)}\, H_\kappa(y;\alpha) = 2\sum_{i=1}^N \binom{\kappa}{\kappa_{(i)}}\, H_{\kappa_{(i)}}(y;\alpha)$$

Proof Applying $E_0^{(y)}$ to the generating function (3.2a) and using (3.3) gives

$$\begin{aligned} \sum_\kappa \frac{1}{|\kappa|!} E_0^{(y)}\, H_\kappa(y;\alpha) C_\kappa^{(\alpha)}(z) &= 2p_1(z)\,{}_0\mathcal{F}_0^{(\alpha)}(2y;z)e^{-p_2(z)} \\ &= 2p_1(z) \sum_\kappa \frac{1}{|\kappa|!}\, H_\kappa(y;\alpha) C_\kappa^{(\alpha)}(z) \qquad (3.13) \end{aligned}$$

Using the formula (3.4) the final formula on the r.h.s. of (3.13) can be rewritten as

$$2\sum_\kappa \frac{1}{(1+|\kappa|)!} \sum_{i=1}^N \binom{\kappa^{(i)}}{\kappa} H_\kappa(y;\alpha) C_{\kappa^{(i)}}^{(\alpha)}(z) = 2\sum_\kappa \frac{1}{|\kappa|!} \sum_{i=1}^N \binom{\kappa}{\kappa_{(i)}}\, H_{\kappa_{(i)}}(y;\alpha) C_\kappa^{(\alpha)}(z) \quad (3.14)$$

Equating coefficients of $C_\kappa^{(\alpha)}(z)$ on the l.h.s. of (3.13) and on the r.h.s. of (3.14) gives the stated result.

Proposition 3.5 *We have*

$$2p_1(y) H_\kappa(y;\alpha) = \alpha \sum_i \binom{\kappa^{(i)}}{\kappa} \frac{j_\kappa}{j_{\kappa^{(i)}}}\, (N-i+1+\alpha\kappa_i)\, H_{\kappa^{(i)}}(y) + 2\sum_i \binom{\kappa}{\kappa_{(i)}}\, H_{\kappa_{(i)}}(y)$$

Proof From the generating function (3.2a) we have

$$\begin{aligned} 2\sum_\kappa \frac{1}{|\kappa|!} p_1(y) H_\kappa(y;\alpha) C_\kappa^{(\alpha)}(z) &= 2p_1(y)\,{}_0\mathcal{F}_0^{(\alpha)}(2y;z)e^{-p_2(z)} = \left(E_0^{(z)}\,{}_0\mathcal{F}_0^{(\alpha)}(2y;z)\right)e^{-p_2(z)} \\ &= \left(E_0^{(z)}\,{}_0\mathcal{F}_0^{(\alpha)}(2y;z)e^{-p_2(z)}\right) + 2p_1(z)\,{}_0\mathcal{F}_0^{(\alpha)}(2y;z)e^{-p_2(z)} \\ &= \left(E_0^{(z)} + 2p_1(z)\right) \sum_\kappa \frac{1}{|\kappa|!} H_\kappa(y;\alpha) C_\kappa^{(\alpha)}(z) \qquad (3.15) \end{aligned}$$

Using the formula (2.13a) and writing

$$2p_1(z)\sum_\kappa \frac{1}{|\kappa|!}H_\kappa(y;\alpha)C_\kappa^{(\alpha)}(z)$$

as in the proof of Proposition 3.4 shows that we can rewrite the last expression on the r.h.s. of (3.15) as

$$\sum_\kappa \frac{1}{|\kappa|!}C_\kappa^\alpha(1^N)\sum_{i=1}^N \binom{\kappa}{\kappa_{(i)}} H_\kappa(y;\alpha)\,\frac{C_{\kappa_{(i)}}^{(\alpha)}(z)}{C_{\kappa_{(i)}}^{(\alpha)}(1^N)} + 2\sum_\kappa \frac{1}{|\kappa|!}\sum_{i=1}^N \binom{\kappa}{\kappa_{(i)}} H_{\kappa_{(i)}}(y;\alpha)C_\kappa^{(\alpha)}(z)$$

$$= \sum_\kappa \frac{1}{(1+|\kappa|)!}\sum_{i=1}^N \binom{\kappa}{\kappa^{(i)}} H_{\kappa^{(i)}}(y;\alpha)\,\frac{C_{\kappa^{(i)}}^{(\alpha)}(1^N)}{C_\kappa^{(\alpha)}(1^N)}C_\kappa^{(\alpha)}(z) + 2\sum_\kappa \frac{1}{|\kappa|!}\sum_{i=1}^N \binom{\kappa}{\kappa_{(i)}} H_{\kappa_{(i)}}(y;\alpha)C_\kappa^{(\alpha)}(z)$$

The stated formula now follows by equating coefficients of $C_\kappa^{(\alpha)}(z)$ on the l.h.s. of (3.15) and the r.h.s. of the above equation, and using (2.15)) to rewrite $C_{\kappa^{(i)}}^{(\alpha)}(1^N)/C_\kappa^{(\alpha)}(1^N)$.

Another consequence of the generating function (3.2a) relates to an analogue of the formula (2.8).

Proposition 3.6 *We have*

$$H_k(\frac{1}{\sqrt{N}}p_1(y)) = N^{-k/2}\sum_{|\kappa|=k} H_\kappa(y;\alpha)C_\kappa^{(\alpha)}(1^N).$$

Proof Set $z_1 = \cdots = z_N = c$ in (3.2a) and note that

$$ {}_0\mathcal{F}_0^{(\alpha)}(2y;c,\dots,c) = \sum_\kappa \frac{c^{|\kappa|}}{|\kappa|!}C_\kappa^{(\alpha)}(2y) = e^{2cp_1(y)} \tag{3.16}$$

(the last equality follows from (2.8)) to conclude

$$\sum_\kappa \frac{c^{|\kappa|}}{|\kappa|!}H_\kappa(y;\alpha)C_\kappa^{(\alpha)}(1^N) = e^{2cp_1(y)}e^{-Nc^2} = \sum_{k=0}^\infty \frac{c^k}{N^{k/2}\,k!}\,H_k(\frac{1}{\sqrt{N}}p_1(y)).$$

The result now follows by equating coefficients of c^k.

Notice that each term on the r.h.s. of the above formula is an eigenfunction of the operator (3.5) with eigenvalue $-2|\kappa|$. Thus $H_k(\frac{1}{\sqrt{N}}p_1(y))$ is also an eigenfunction of (3.5) with eigenvalue $-2k$. This latter fact can be checked directly, and has been observed previously [11].

3.3 Integration formulas

Using the generating function (3.2a) and the orthogonality of the generalized Hermite polynomials with respect to the inner product (2.20), a number of integration formulas can be obtained. In particular, we can obtain the multidimensional analogues of the classical formulas

$$\int_{-\infty}^\infty dy\, e^{-y^2}\Big(H_k(y)\Big)^2 = \sqrt{\pi}2^k k! \tag{3.17a}$$

$$\frac{2^{-k}}{\sqrt{\pi}}\int_{-\infty}^\infty dy\, e^{-y^2}H_k(y+x) = x^k \tag{3.17b}$$

$$\frac{2^k}{\sqrt{\pi}}\int_{-\infty}^\infty dy\, e^{-y^2}(x+iy)^k = H_k(x). \tag{3.17c}$$

To present these analogues let us introduce the notation

$$d\mu^{(H)}(y) := \prod_{j=1}^{N} e^{-y_j^2} \prod_{1 \le j < k \le N} |y_j - y_k|^{2/\alpha}\, dy_1 \dots dy_N. \tag{3.18}$$

Proposition 3.7 *We have*

$$\mathcal{N}_\kappa^{(H)} := \int_{(-\infty,\infty)^N} \Big(H_\kappa(y;\alpha)\Big)^2 d\mu^{(H)}(y) = \frac{2^{|\kappa|}|\kappa|!\mathcal{N}_0^{(H)}}{C_\kappa^{(\alpha)}(1^N)}$$

where

$$\mathcal{N}_0^{(H)} := \int_{(-\infty,\infty)^N} d\mu^{(H)}(y) = 2^{-N(N-1)/2\alpha}\pi^{N/2} \prod_{j=0}^{N-1} \frac{\Gamma(1+(j+1)/\alpha)}{\Gamma(1+1/\alpha)}.$$

Proof Multiplying both sides of the generating function (3.2a) by $H_\kappa(y;\alpha)$, integrating with respect to the measure (3.18), and using the orthogonality property of $\{H_\kappa(y;\alpha)\}_\kappa$ with respect to the inner product (2.20) gives

$$\frac{\mathcal{N}_\kappa^{(H)}}{|\kappa|!} C_\kappa^{(\alpha)}(z) = e^{-p_2(z)} \int_{(-\infty,\infty)^N} {}_0\mathcal{F}_0^{(\alpha)}(2y;z) H_\kappa(y;\alpha)\, d\mu^{(H)}(y). \tag{3.19}$$

Set $z_1 = \dots = z_N = c$, substitute (3.16) in the r.h.s. of (3.19) and complete the square to show

$$\frac{\mathcal{N}_\kappa^{(H)}}{|\kappa|!} C_\kappa^{(\alpha)}(c) = \int_{(-\infty,\infty)^N} e^{-p_2(y)} \prod_{1 \le j < k \le N} |y_k - y_j|^{2/\alpha} H_\kappa(y+c;\alpha)\, dy_1 \dots dy_N.$$

Now take the limit $c \to \infty$. Since from (2.14) and (2.11a)

$$\lim_{c\to\infty} \frac{H_\kappa(y+c;\alpha)}{C_\kappa^{(\alpha)}(c)} = \frac{2^{|\kappa|}}{C_\kappa^{(\alpha)}(1^N)}$$

the stated formula for $\mathcal{N}_\kappa^{(H)}$ follows. The formula for $\mathcal{N}_0^{(H)}$ is a well known limiting case of Selberg's integral.

The analogues of (3.17b) and (3.17c) can be derived from the following integration formula.

Proposition 3.8 *We have*

$$\int_{(\infty,\infty)^N} {}_0\mathcal{F}_0^{(\alpha)}(2y;z) {}_0\mathcal{F}_0^{(\alpha)}(2y;w)\, d\mu(y) = e^{p_2(w)+p_2(z)}\, \mathcal{N}_0^{(H)}\, {}_0\mathcal{F}_0^{(\alpha)}(2z;w)$$

Proof Substitute the generating function (3.2a) for ${}_0\mathcal{F}_0^{(\alpha)}$ and integrate term-by-term using the orthogonality property of the $\{H_\kappa(y;\alpha)\}_\kappa$ with respect to the inner product (2.20) and the normalization integral of Proposition 3.7. The resulting series is identified as ${}_0\mathcal{F}_0^{(\alpha)}(2z;w)$ according to the definition (3.2b).

Corollary 3.1 *We have*

$$\int_{(-\infty,\infty)^N} {}_0\mathcal{F}_0^{(\alpha)}(2y;z) H_\kappa(y,\alpha)\, d\mu(y) = e^{p_2(z)}\, \mathcal{N}_0^{(H)} \frac{2^{|\kappa|} C_\kappa^{(\alpha)}(z)}{C_\kappa^{(\alpha)}(1^N)}$$

Proof Multiply both sides of the integration formula of Proposition 3.8 by $\exp(-p_2(w))$ and substitute for $\exp(-p_2(w)) {}_0\mathcal{F}_0^{(\alpha)}(2y;w)$ using (3.2a). The result follows by equating coefficients of $C_\kappa^{(\alpha)}(w)$ on both sides.

Corollary 3.2 *We have*

$$e^{-p_2(z)}\, H_\kappa(z;\alpha) = \frac{2^{|\kappa|}}{\mathcal{N}_0^{(H)} C_\kappa^{(\alpha)}(1^N)} \int_{(\infty,\infty)^N} {}_0\mathcal{F}_0^{(\alpha)}(2y;-iz) C_\kappa^{(\alpha)}(iy)\, d\mu(y)$$

Proof By writing iw for w in Proposition 3.8, we can replace ${}_0\mathcal{F}_0^{(\alpha)}(2y;w)$ by ${}_0\mathcal{F}_0^{(\alpha)}(2iy;w)$, ${}_0\mathcal{F}_0^{(\alpha)}(2z;w)$ by ${}_0\mathcal{F}_0^{(\alpha)}(2iz;w)$ and $\exp(-p_2(w))$ by $\exp(p_2(w))$. The result follows by using the generating function (3.2a) to substitute for $\exp(-p_2(w)) {}_0\mathcal{F}_0^{(\alpha)}(2iz;w)$ and equating coefficients of $C_\kappa^{(\alpha)}(w)$.

The integration formula of Corollary 3.2 can be used to derive the analogue of the classical summation formula [8]

$$\sum_{k=0}^\infty \frac{H_k(w)H_k(z)}{k!2^k\sqrt{\pi}}\, t^k = \frac{1}{\sqrt{\pi}}(1-t^2)^{-1/2} e^{-t^2(z^2+w^2)/(1-t^2)} e^{2wzt/(1-t^2)}, \quad |t|<1.$$

Proposition 3.9 *For $|t|<1$ we have*

$$\begin{aligned} G^{(H)}(w,z;t) &:= \sum_\kappa \frac{H_\kappa(w;\alpha)H_\kappa(z;\alpha)}{\mathcal{N}_\kappa^{(H)}} t^{|\kappa|} \\ &= \frac{1}{\mathcal{N}_0^{(H)}}(1-t^2)^{-Nq/2}\ \exp\left(-\frac{t^2}{(1-t^2)}\left(p_2(z)+p_2(w)\right)\right) \\ &\quad \times {}_0\mathcal{F}_0^{(\alpha)}\left(\frac{2wt}{(1-t^2)^{1/2}}; \frac{z}{(1-t^2)^{1/2}}\right) \end{aligned}$$

where $q = 1 + (N-1)/\alpha$.

Proof Substituting the integral representation of Corollary 3.2 for $H_\kappa(z;\alpha)$ and $H_\kappa(w;\alpha)$ in the definition $G^{(H)}(w,z;t)$, we see that the sum over κ can be recognized in terms of ${}_0\mathcal{F}_0^{(\alpha)}$ and thus

$$\begin{aligned} G^{(H)}(w,z;t) &= e^{p_2(z)+p_2(w)} \frac{1}{(\mathcal{N}_0^{(H)})^3} \\ &\times \int_{(-\infty,\infty)^N} d\mu^{(H)}(y_a) \int_{(-\infty,\infty)^N} d\mu^{(H)}(y_b) {}_0\mathcal{F}_0^{(\alpha)}(2y_a;-iw) {}_0\mathcal{F}_0^{(\alpha)}(2y_b;-iz) {}_0\mathcal{F}_0^{(\alpha)}(2y_a;-ty_b) \end{aligned}$$

We now use Proposition 3.8 to integrate over y_a. This gives

$$\begin{aligned} G^{(H)}(w,z;t) &= e^{p_2(z)} \frac{1}{(\mathcal{N}_0^{(H)})^2} \int_{(-\infty,\infty)^N} d\mu^{(H)}(y_b) {}_0\mathcal{F}_0^{(\alpha)}(2y_b;-iz) {}_0\mathcal{F}_0^{(\alpha)}(2iw;ty_b)\, e^{t^2 p_2(y_b)} \\ &= e^{p_2(z)} \frac{1}{(\mathcal{N}_0^{(H)})^2} (1-t^2)^{-(N/2+N(N-1)/2\alpha)} \\ &\times \int_{(-\infty,\infty)^N} d\mu^{(H)}(y_b) {}_0\mathcal{F}_0^{(\alpha)}(2y_b;-iz(1-t^2)^{-1/2}) {}_0\mathcal{F}_0^{(\alpha)}(2y_b;iwt(1-t^2)^{-1/2}) \end{aligned}$$

where the second equality follows by combining $d\mu^{(H)}(y_b)$ and $\exp(t^2 p_2(y_b))$ (recall (3.18)) and changing variables. The integration over $d\mu^{(H)}(y_b)$ can now be performed using Proposition 3.8, and the summation formula for $G^{(H)}(w,z;t)$ results.

Notice from (3.16) that in the special case that $w_1 = \cdots = w_N = c$, the summation formula of Proposition 3.9 is entirely in terms of elementary functions:

$$G^{(H)}(w,z;t) = \frac{1}{\mathcal{N}_0^{(H)}}(1-t^2)^{-Nq/2} \exp\left(-\frac{1}{(1-t^2)} \sum_{j=1}^N (t^2 z_j^2 - 2tcz_j + t^2c^2) \right). \tag{3.20}$$

Interpretation of this result in terms of an explicit solution of the Fokker-Planck equation (1.6) with W given by (1.3a) will be discussed in Section 5.

In Corollary 3.2 a certain integral transform is applied to the Jack polynomial to obtain the generalized Hermite polynomial. It has been observed by Lassalle that the generalized Hermite polynomials can be obtained from the Jack polynomials by the action of a certain exponential differential operator. Thus from the formula (3.7b), we see that

$$\Big(\frac{1}{4}D_0^{(y)}\Big)^k {}_0\mathcal{F}_0^{(\alpha)}(2y;z) = \Big(p_2(z)\Big)^k {}_0\mathcal{F}_0^{(\alpha)}(2y;z),$$

which after multiplication by $(-1)^k/k!$ and summing over k gives

$$\exp\Big(-\frac{1}{4}D_0^{(y)}\Big) {}_0\mathcal{F}_0^{(\alpha)}(2y;z) = e^{-p_2(z)} {}_0\mathcal{F}_0^{(\alpha)}(2y;z).$$

Use of the generating function (3.2a) on the r.h.s. and equating coefficients of $C_\kappa^{(\alpha)}(z)$ gives Lassalle's formula

$$\frac{2^{|\kappa|}}{C_\kappa^{(\alpha)}(1^N)} \exp\Big(-\frac{1}{4}D_0^{(y)}\Big) C_\kappa^{(\alpha)}(y) = H_\kappa(y;\alpha). \tag{3.21}$$

Comparison with the formula of Corollary 3.2, and use of the fact that $\{C_\kappa^{(\alpha)}(y)\}_\kappa$ forms a basis for symmetric analytic functions shows that for any symmetric analytic function $f(y)$,

$$\frac{e^{p_2(z)}}{\mathcal{N}_0^{(H)}} \int_{(-\infty,\infty)^N} {}_0\mathcal{F}_0^{(\alpha)}(2y;-iz) f(iy)\, d\mu^{(H)}(y) = \exp\Big(-\frac{1}{4}D_0^{(z)}\Big) f(z). \tag{3.22}$$

From (3.22) we see that if

$$F(z) = \frac{e^{p_2(z)}}{\mathcal{N}_0^{(H)}} \int_{(-\infty,\infty)^N} {}_0\mathcal{F}_0^{(\alpha)}(2y;-iz) f(iy)\, d\mu^{(H)}(y) \tag{3.23a}$$

then

$$f(z) = \exp\Big(\frac{1}{4}D_0^{(z)}\Big) F(z). \tag{3.23b}$$

On the other hand, by replacing z by iz and $f(ix)$ by $F(x)$ we have

$$\frac{e^{-p_2(z)}}{\mathcal{N}_0^{(H)}} \int_{(-\infty,\infty)^N} {}_0\mathcal{F}_0^{(\alpha)}(2y;z) F(y)\, d\mu^{(H)}(y) = \exp\Big(\frac{1}{4}D_0^{(z)}\Big) F(z). \tag{3.24}$$

Comparison of (3.23b) and (3.24) gives

$$f(z) = \frac{e^{-p_2(z)}}{\mathcal{N}_0^{(H)}} \int_{(-\infty,\infty)^N} {}_0\mathcal{F}_0^{(\alpha)}(2y; z) F(y)\, d\mu^{(H)}(y) \tag{3.25}$$

which is the inversion formula for the transform (3.23a) (in the case $\alpha = \infty$ (3.23a) corresponds to the Fourier transform).

4 The generalized Laguerre polynomials

The generalized Laguerre polynomials, defined as the polynomial eigenfunctions of the operator (2.1b) of the form (2.11b) with normalization (2.16), also satisfy higher-dimensional analogues of their classical counterparts. A number of these formulas have been proved in the case $\alpha = 2$ by Muirhead [27] and for general α but $N = 2$ by Yan [34]. Below we will develop the theory of generalized Laguerre polynomials by presenting the analogues of the classical generating functions, the series expansion (2.2b), recurrence and differentiation formulas, integration formulas and a summation formula. In Section 6 we will identify the formulas known to Muirhead and Yan, as well as those which can be found in the work of Lassalle and Macdonald.

4.1 Generating functions

The classical Laguerre polynomials can be defined by either of the generating functions

$$e^z \frac{J_a(2\sqrt{yz})}{(yz)^{a/2}} = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+a+1)} L_n^a(y) z^n \tag{4.1a}$$

or

$$(1-z)^{-(a+1)} e^{yz/(z-1)} = \sum_{n=0}^{\infty} L_n^a(y) z^n \tag{4.1b}$$

where in (4.1a) J_a denotes the Bessel function. These generating functions have the following higher-dimensional analogues.

Proposition 4.1 *We have*

$$e^{p_1(z)} {}_0\mathcal{F}_1^{(\alpha)}(a+q; x; -z) = \sum_\kappa \frac{L_\kappa^a(x;\alpha) C_\kappa^{(\alpha)}(z)}{[a+q]_\kappa^{(\alpha)}} \tag{4.2}$$

where

$$q := 1 + (N-1)/\alpha, \tag{4.3a}$$

$${}_p\mathcal{F}_r^{(\alpha)}(a_1, \dots, a_p; b_1, \dots, b_r; x; z) := \sum_\kappa \frac{1}{|\kappa|!} \frac{[a_1]_\kappa^{(\alpha)} \dots [a_p]_\kappa^{(\alpha)}}{[b_1]_\kappa^{(\alpha)} \dots [b_r]_\kappa^{(\alpha)}} \frac{C_\kappa^{(\alpha)}(x) C_\kappa^{(\alpha)}(z)}{C_\kappa^{(\alpha)}(1^N)} \tag{4.3b}$$

with $[c]_\kappa^{(\alpha)} := \prod_{j=1}^N \left(c - \frac{1}{\alpha}(j-1)\right)_{\kappa_j}$. *We also have*

$$\left(\prod (1-z)\right)^{-(a+q)} {}_0\mathcal{F}_0^{(\alpha)}(-x; \frac{z}{1-z}) = \sum_\kappa L_\kappa^a(x;\alpha) C_\kappa^{(\alpha)}(z) \tag{4.4}$$

where $\prod(1-z) := \prod_{j=1}^N (1-z_j)$.

Proof In each case we need to establish that $L_\kappa^a(x;\alpha)$ as defined by the generating function is an eigenfunction of the operator (2.1b), which in terms of the notation (2.12) reads

$$\tilde{H}^{(L)} = D_1 + (a+1)E_0 - E_1, \tag{4.5}$$

with eigenvalue $-|\kappa|$ and has an expansion in terms of Jack polynomials with highest weight term $(-1)^{|\kappa|}C_\kappa^{(\alpha)}(x)/|\kappa|!C_\kappa^\alpha(1^N)$. The proof of the first requirement relies on the identities

$$\Big(D_1^{(x)} + (a+1)E_0^{(x)}\Big){}_0\mathcal{F}_1^{(\alpha)}(a+q;x;z) = p_1(z){}_0\mathcal{F}_1^{(\alpha)}(a+q;x;z) \tag{4.6a}$$

$$\Big(D_1^{(x)} - E_2^{(y)}\Big){}_0\mathcal{F}_0^{(\alpha)}(x;y) = (q-1)p_1(y){}_0\mathcal{F}_0^{(\alpha)}(x;y) \tag{4.6b}$$

with (4.6a) being established in the Appendix, and (4.6b) simply a rewrite of (3.7a) with $y \to y/2$.

First consider (4.2). We have

$$E_1^{(z)}{}_0\mathcal{F}_1^{(\alpha)}(a+q;x;-z)e^{p_1(z)} = e^{p_1(z)}E_1^{(z)}{}_0\mathcal{F}_1^{(\alpha)}(a+q;x;-z) + p_1(z)e^{p_1(z)}{}_0\mathcal{F}_1^{(\alpha)}(a+q;x;-z). \tag{4.7}$$

Using (4.6a) and the fact that $E_1^{(z)}$ is an eigenoperator of the Jack polynomials so that its action on ${}_0\mathcal{F}_1^{(\alpha)}(x;-z)$ is the same as the action of $E_1^{(y)}$, the r.h.s. of (4.7) can be rewritten as

$$\Big(E_1^{(x)} - D_1^{(x)} - (a+1)E_0^{(x)}\Big){}_0\mathcal{F}_1^{(\alpha)}(a+q;x;-z)e^{p_1(z)}. \tag{4.8}$$

Substituting the generating function (4.2) in the l.h.s. of (4.7) and computing the action of $E_1^{(z)}$, and comparing coefficients of $C_\kappa^{(\alpha)}(z)$ with (4.8) after also substituting the generating function (4.2) establishes the eigenvalue equation. The expansion of $L_\kappa^{(\alpha)}(x;\alpha)$ in terms of Jack polynomials as deduced from (4.2) is given by the Proposition 4.3. Its highest weight term is $(-1)^{|\kappa|}C_\kappa^{(\alpha)}(x)/C_\kappa^{(\alpha)}(1^N)$ as required.

Now consider (4.4). Setting $y_j := z_j/(1-z_j)$ this is equivalent to

$$\sum_\mu L_\mu^a(x;\alpha)C_\mu^{(\alpha)}(\frac{y}{1+y}) = \Big(\prod(1+y)\Big)^{a+q}{}_0F_0^{(\alpha)}(-x;y) \tag{4.9}$$

To verify the eigenvalue equation, note that $E_1^{(y)} + E_2^{(y)}$ is an eigenoperator of $C_\mu^{(\alpha)}(y/(1+y))$ with eigenvalue $|\mu|$, and

$$(E_1^{(y)} + E_2^{(y)})\Big(\prod(1+y)\Big)^{a+q} = (a+q)p_1(y)\Big(\prod(1+y)\Big)^{a+q}. \tag{4.10}$$

Thus, applying $E_1^{(y)} + E_2^{(y)}$ to the generating function (4.9) we have

$$\begin{aligned}
&\sum_\mu |\mu| L_\mu^a(x;\alpha)C_\mu^{(\alpha)}(\frac{y}{1+y}) \\
&= E_1^{(y)} + E_2^{(y)})\Big(\prod(1+y)\Big)^{a+q}{}_0\mathcal{F}_0^{(\alpha)}(-x;y) \\
&= \Big(\prod(1+y)\Big)^{a+q}\Big(E_1^{(y)} + E_2^{(y)} + (a+q)p_1(y)\Big){}_0\mathcal{F}_0^{(\alpha)}(-x;y) \\
&= \Big(\prod(1+y)\Big)^{a+q}\Big(E_1^{(x)} - D_1^{(x)} + (a+1)p_1(y)\Big){}_0\mathcal{F}_0^{(\alpha)}(-x;y) \\
&= \Big(\prod(1+y)\Big)^{a+q}\Big(E_1^{(x)} - D_1^{(x)} - (a+1)E_0^{(x)}\Big){}_0\mathcal{F}_0^{(\alpha)}(-x;y)
\end{aligned} \tag{4.11}$$

where to obtain the third equality we have used (4.6b) while the final equality follows from (3.3). The factor involving $\prod(1+y)$ in the final equality can be commuted in front of the operators, since they act only on the x-variables. Use of the generating function (4.9) and comparison of the coefficient of $C_\kappa^{(\alpha)}(y/(1+y))$ in (4.11) establishes the eigenvalue equation. The highest weight term in the Jack polynomial expansion of $L_\kappa^a(x;\alpha)$ as defined by (4.4) is the same as the highest weight coefficient of $C_\kappa^{(\alpha)}(z)$ in the expansion of

$$\Big(\prod(1-z)\Big)^{-(a+q)} \sum_{\substack{\mu \\ |\mu|\le|\kappa|}} \frac{(-1)^{|\mu|}}{|\mu|!} C_\mu^{(\alpha)}(x) \frac{C_\mu^{(\alpha)}(z/(1-z))}{C_\mu^{(\alpha)}(1^N)}$$

Consideration of the form of the expansion of $C_\mu^{(\alpha)}(z/(1-z))$ and $\Big(\prod(1-z)\Big)^{-(a+q)}$ in terms of $\{C_\sigma^{(\alpha)}(z)\}_\sigma$ shows that the highest weight coefficient of $C_\kappa^{(\alpha)}(z)$ in the above expression is $(-1)^{|\kappa|}C_\kappa^{(\alpha)}(x)/|\kappa|!C_\kappa^{(\alpha)}(1^N)$. Comparison with the coefficient of $C_\kappa^{(\alpha)}(z)$ on the r.h.s. of (4.4) shows that $L_\kappa^{(\alpha)}(x;\alpha)$ has the required highest weight term for its expansion in terms of Jack polynomials.

In Proposition 4.1 the Laguerre polynomial is given by generating functions involving ${}_0\mathcal{F}_0^{(\alpha)}$ and ${}_0\mathcal{F}_1^{(\alpha)}$. It is also possible to give a generating function involving ${}_1\mathcal{F}_1^{(\alpha)}$ (recall (4.3b)) which includes both these generating functions as limiting cases.

Proposition 4.2 *We have*

$$\prod(1-z)^{-c-q}\, {}_1\mathcal{F}_1^{(\alpha)}\Big(c+q;a+q;-x;\frac{z}{1-z}\Big) = \sum_\lambda \frac{[c+q]_\lambda^{(\alpha)}}{[a+q]_\lambda^{(\alpha)}}\, L_\lambda^a(x;\alpha)\, C_\lambda^{(\alpha)}(z). \tag{4.12}$$

Proof The derivation closely follows that of (4.4) above, with (4.6b) being replaced by the formula

$$\Big(D_1^{(x)} + (a+\frac{1-N}{\alpha})E_0^{(x)}\Big)\, {}_1\mathcal{F}_1^{(\alpha)}(c;a;x;y) = \Big(E_2^{(y)} + cp_1(y)\Big)\, {}_1\mathcal{F}_1^{(\alpha)}(c;a;x;y) \tag{4.13}$$

which is established in the Appendix.

To derive the generating function (4.2) from (4.12) replace z by z/c and take the limit $c\to\infty$ using the facts that

$$\lim_{c\to\infty} {}_1\mathcal{F}_1^{(\alpha)}(c+q;a+q;-x;\frac{z/c}{1-z/c}) = {}_0\mathcal{F}_1^{(\alpha)}(a+q;-x;z),$$

$$\lim_{c\to\infty}\prod(1-z/c)^{-c-q} = e^{p_1(z)}, \quad \lim_{c\to\infty}[c+q]_\kappa^{(\alpha)}C_\kappa^{(\alpha)}(z/c) = C_\kappa^{(\alpha)}(z).$$

The generating function (4.4) follows from (4.12) by setting $c=a$.

From the generating function (4.2) it is possible to deduce the higher-dimensional analogue of the series expansion (2.2b).

Proposition 4.3 *We have*

$$L_\kappa^a(x;\alpha) = \frac{[a+q]_\kappa^{(\alpha)}}{|\kappa|!}\sum_{\sigma\subseteq\kappa}\binom{\kappa}{\sigma}\frac{(-1)^{|\sigma|}C_\sigma^{(\alpha)}(x)}{[a+q]_\sigma^{(\alpha)}C_\sigma^{(\alpha)}(1^N)} \tag{4.14a}$$

$$C_\kappa^{(\alpha)}(x) = [a+q]_\kappa^{(\alpha)}C_\kappa^{(\alpha)}(1^N)\sum_{\sigma\subseteq\kappa}(-1)^{|\sigma|}\binom{\kappa}{\sigma}\frac{|\sigma|!L_\sigma^a(x;\alpha)}{[a+q]_\sigma^{(\alpha)}} \tag{4.14b}$$

Proof The formula (4.14a) follows from the generating function formula (4.2) by applying the identity [20, 17]

$$e^{p_1(z)}\, C_\lambda^{(\alpha)}(z) = \sum_\mu \binom{\mu}{\lambda} \frac{|\lambda|!}{|\mu|!} C_\mu^{(\alpha)}(z) \tag{4.15}$$

on the l.h.s. and equating coefficients of $C_\kappa^{(\alpha)}(z)$. The formula (4.14b) follows from (4.2) by multiplying both sides by $e^{-p_1(z)}$, using the identity (4.15) (with z replaced by $-z$) on the r.h.s., and equating coefficients of $C_\kappa^{(\alpha)}(z)$.

A simple consequence of the generating function (4.4) is the analogue of the formula of Proposition 3.6, which is derived by following the steps of the proof of that formula.

Proposition 4.4 *We have*

$$L_k^{N(a+q)-1}(p_1(y)) = \sum_{|\kappa|=k} L_\kappa^a(y;\alpha) C_\kappa^{(\alpha)}(1^N).$$

Since each $L_\kappa^a(y;\alpha)$ is an eigenfunction of (4.5) with eigenvalue $-|\kappa|$, it follows that $L_k^{N(a+q)-1}(p_1(y))$ is also an eigenfunction of (4.5) with eigenvalue $-k$ (this feature can be checked directly).

As our final result of this section, we note that the proof of (4.2) can be generalized to give a family of eigenoperators for $L_\kappa^a(y;\alpha)$, together with the corresponding eigenvalues. These operators are analogues of the operators of Proposition 3.2 for the generalized Hermite polynomials, and are derived in the same way.

Proposition 4.5 *Let*

$$\begin{aligned}\tilde H^{(L)}(y) := \Big(D_N^{p\,(y)} - \Big[D_1^{(y)} + (a+1)E_0^{(y)}, D_N^{p\,(y)}\Big] + \cdots \\ + \frac{(-1)^n}{n!}\Big[D_1^{(y)} + (a+1)E_0^{(y)}, \Big[\cdots \Big[D_1^{(y)} + (a+1)E_0^{(y)}, D_N^{p\,(y)}\Big]\cdots\Big]\Big]\Big)\end{aligned}$$

where D_N^p is the operator introduced in Section 3.1. We have that $L_\kappa^a(y;\alpha)$ is an eigenfunction of $\tilde H^{(L)}(y)$ for each $p = 1, \dots, N$, with eigenvalue $e_p(\kappa;\alpha)$ given by the coefficient of X^{N-p} in (3.10).

4.2 Recurrence and differentiation formulas

The classical Laguerre polynomials satisfy the recurrence relations

$$xL_n^a(x) = (2n+a+1)L_n^a(x) - (n+1)L_{n+1}^a(x) - (n+a)L_{n-1}^a(x) \tag{4.16a}$$

$$L_n^{a+1}(x) = \sum_{m=0}^n L_m^a(x) \tag{4.16b}$$

$$L_n^a(x) = L_n^{a+1}(x) - L_n^{a+1}(x) \tag{4.16c}$$

and the differentiation formulas

$$\frac{d}{dx}L_n^a(x) = -L_{n-1}^{a+1}(x) \tag{4.17a}$$

$$x\frac{d}{dx}L_n^a(x) = nL_n^a(x) - (n+a)L_{n-1}^a(x). \tag{4.17b}$$

The generalized Laguerre polynomials satisfy higher-dimensional analogues of these formulas. Let us first consider (4.17b).

Proposition 4.6 *We have*

$$E_1^{(x)}\, L_\kappa^a(x;\alpha) = |\kappa| L_\kappa^a(x;\alpha) - \frac{1}{|\kappa|}\sum_i \binom{\kappa}{\kappa_{(i)}} \left(\kappa_i + a + \frac{N-i}{\alpha}\right) L_{\kappa_{(i)}}^a(x;\alpha)$$

Proof From the generating function (4.9) and the fact that $E_1^{(x)}$ is an eigenoperator of $C_\kappa^{(\alpha)}(x)$ we have

$$\begin{aligned}
&E_1^{(x)} \sum_\mu L_\mu^a(x;\alpha) C_\mu^{(\alpha)}(\frac{y}{1+y}) \\
&= \prod (1+y)^{a+q} E_1^{(y)}\, {}_0\mathcal{F}_0^{(\alpha)}(-x;y) \\
&= \left(E_1^{(y)} - (a+q)p_1(y/(1+y))\right) \prod (1+y)^{a+q}\, {}_0\mathcal{F}_0^{(\alpha)}(-x;y) \\
&= \left(E_1^{(y)} - (a+q)p_1(y/(1+y))\right) \sum_\mu L_\mu^a(x;\alpha) C_\mu^{(\alpha)}(\frac{y}{1+y})
\end{aligned} \tag{4.18}$$

But with $z_j := y_j/(1+y_j)$,

$$\begin{aligned}
E_1^{(y)} C_\mu^{(\alpha)}(\frac{y}{1+y}) &= (E_1^{(z)} - E_2^{(z)}) C_\mu^{(\alpha)}(z) \\
&= |\mu| C_\mu^{(\alpha)}(z) - \frac{1}{1+|\mu|} \sum_{i=1}^N \binom{\mu^{(i)}}{\mu} \left(\mu_i - \frac{i-1}{\alpha}\right) C_{\mu^{(i)}}^{(\alpha)}(z)
\end{aligned}$$

where the second equality uses (2.13c). Substituting this expression in the r.h.s. of (4.18), and using (3.4) to simplify the remaining term on the r.h.s. of (4.18) gives

$$\begin{aligned}
E_1^{(x)} \sum_\mu L_\mu^a(x;\alpha) C_\mu^{(\alpha)}(\frac{y}{1+y}) = \sum_\mu L_\mu^a(x;\alpha) \Bigg\{ |\mu| C_\mu^{(\alpha)}(\frac{y}{1+y}) \\
- \frac{1}{1+|\mu|} \sum_{i=1}^N \binom{\mu^{(i)}}{\mu} \left(\mu_i + 1 + a + \frac{N-i}{\alpha}\right) C_{\mu^{(i)}}^{(\alpha)}(\frac{y}{1+y}) \Bigg\}
\end{aligned}$$

The result follows by equating coefficients of $C_\mu^{(\alpha)}(\frac{y}{1+y})$.

Next we will derive the analogue of (4.16a).

Proposition 4.7 *We have*

$$\begin{aligned}
p_1(x)\, L_\kappa^a(x;\alpha) = (2|\kappa| + N(a+q))\, L_\kappa^a(x;\alpha) + \frac{1}{|\kappa|} \sum_{i=1}^N \binom{\kappa}{\kappa_{(i)}} \left(\kappa_i + a + \frac{N-i}{\alpha}\right) L_{\kappa_{(i)}}^a(x;\alpha) \\
-(|\kappa|+1)\alpha \sum_i \binom{\kappa^{(i)}}{\kappa} \frac{j_\kappa}{j_{\kappa^{(i)}}} (N - i + 1 + \alpha\kappa_i)\; L_{\kappa^{(i)}}^a(x;\alpha)
\end{aligned}$$

Proof From the generating function (4.9)

$$\sum_\mu p_1(x) L_\mu^a(x;\alpha) C_\mu^{(\alpha)}(\frac{y}{1+y}) = p_1(x) \prod (1+y)^{a+q}\, {}_0\mathcal{F}_0^{(\alpha)}(-x;y)$$

$$
\begin{aligned}
&= -\prod(1+y)^{a+q}E_0^{(y)}\,{}_0\mathcal{F}_0^{(\alpha)}(-x;y) \\
&= -E_0^{(y)}\prod(1+y)^{a+q}\,{}_0\mathcal{F}_0^{(\alpha)}(-x;y) + {}_0\mathcal{F}_0^{(\alpha)}(-x;y)E_0^{(y)}\prod(1+y)^{a+q} \\
&= \Big(-E_0^{(y)} + (a+q)p_1(1/(1+y))\Big)\prod(1+y)^{a+q}\,{}_0\mathcal{F}_0^{(\alpha)}(-x;y) \\
&= \Big(-E_0^{(y)} + (a+q)p_1(1/(1+y))\Big)\sum_\mu L_\mu^a(x;\alpha)C_\mu^{(\alpha)}(\frac{y}{1+y}) \qquad (4.19)
\end{aligned}
$$

The task is now to write $E_0^{(y)}\,C_\mu^{(\alpha)}(\frac{y}{1+y})$ and $p_1(1/(1+y))C_\mu^{(\alpha)}(\frac{y}{1+y})$ as a series in $\{C_\kappa^{(\alpha)}(\frac{y}{1+y})\}_\kappa$. To do this let $z_j = y_j/(1+y_j)$ so that

$$
E_0^{(y)} = \sum_{j=1}^N (1-z_j)^2\frac{\partial}{\partial z_j} = E_0^{(z)} - 2E_1^{(z)} + E_2^{(z)} \quad \text{and} \quad p_1(1/(1+y)) = p_1(1-z) = N - p_1(z).
$$

We then have

$$
\begin{aligned}
E_0^{(y)}\,C_\mu^{(\alpha)}(\frac{y}{1+y}) &= \Big(E_0^{(z)} - 2E_1^{(z)} + E_2^{(z)}\Big)C_\mu^{(\alpha)}(z) \\
&= \sum_{i=1}^N \binom{\mu}{\mu_{(i)}}\frac{C_\mu^{(\alpha)}(1^N)}{C_{\mu_{(i)}}^{(\alpha)}(1^N)}C_{\mu_{(i)}}^{(\alpha)}(\frac{y}{1+y}) - 2|\mu|C_\mu^{(\alpha)}(\frac{y}{1+y}) \\
&\quad + \frac{1}{1+|\mu|}\sum_{i=1}^N \binom{\mu^{(i)}}{\mu}\left(\mu_i - \frac{i-1}{\alpha}\right)C_{\mu^{(i)}}(\frac{y}{1+y}) \qquad (4.20)
\end{aligned}
$$

and

$$
\begin{aligned}
p_1(1/(1+y))C_\mu^{(\alpha)}(\frac{y}{1+y}) &= \Big(N - p_1(z)\Big)C_\mu^{(\alpha)}(z) \\
&= NC_\mu^{(\alpha)}(\frac{y}{1+y}) - \frac{1}{1+|\mu|}\sum_{i=1}^N \binom{\mu^{(i)}}{\mu}C_{\kappa^{(i)}}^{(\alpha)}(\frac{y}{1+y}) \qquad (4.21)
\end{aligned}
$$

where to obtain (4.20) we have used (2.13a), (2.13b) and (2.13c), while to obtain (4.21) we have used (3.4). Substituting (4.20) and (4.21) in the r.h.s. of (4.19), equating coefficients of $C_{\kappa^{(i)}}^{(\alpha)}(\frac{y}{1+y})$ with the l.h.s. of (4.19), and use of (2.15) to rewrite $C_\mu^{(\alpha)}(1^N)/C_{\mu_{(i)}}^{(\alpha)}(1^N)$ gives the stated result.

The generalizations of (4.16b) and (4.16c) are given by the following result.

Proposition 4.8 *We have*

$$
L_\kappa^{a-1}(x;\alpha) = \sum_{r=0}^{\min(N,|\kappa|)}(-\alpha)^r \sum_{\substack{\sigma \\ \kappa/\sigma \text{ a vertical } r\text{-strip}}} \frac{|\sigma|!}{|\kappa|!}\psi_{\kappa/\sigma}(\alpha)L_\sigma^a(x;\alpha) \qquad (4.22)
$$

$$
L_\kappa^{a+1/\alpha}(x;\alpha) = \sum_{r=0}^{|\kappa|}\alpha^{-r} \sum_{\substack{\sigma \\ \kappa/\sigma \text{ a horizontal } r\text{-strip}}} \frac{|\sigma|!}{|\kappa|!}\phi_{\kappa/\sigma}(\alpha)L_\sigma^a(x;\alpha) \qquad (4.23)
$$

where

$$
\psi_{\kappa/\sigma}(\alpha) = \prod_{s\in R_{\kappa/\sigma}\cap\kappa}\frac{h_\kappa^*(s)}{h_*^\kappa(s)}\prod_{s\in R_{\kappa/\sigma}\cap\sigma}\frac{h_*^\sigma(s)}{h_\sigma^*(s)}\prod_{s\in\kappa}h_*^\kappa(s)\prod_{s\in\sigma}(h_*^\sigma(s))^{-1} \qquad (4.24a)
$$

$$
\phi_{\kappa/\sigma}(\alpha) = \prod_{s\in C_{\kappa/\sigma}\cap\kappa}\frac{h_*^\kappa(s)}{h_\kappa^*(s)}\prod_{s\in C_{\kappa/\sigma}\cap\sigma}\frac{h_\sigma^*(s)}{h_*^\sigma(s)}\prod_{s\in\kappa}h_\kappa^*(s)\prod_{s\in\sigma}(h_\sigma^*(s))^{-1} \qquad (4.24b)
$$

Here $R_{\kappa/\sigma}$ denotes the union of all rows which intersect $\kappa-\sigma$, $C_{\kappa/\sigma}$ denotes the union of all columns which intersect $\kappa-\sigma$, and $h_\kappa^*(s)$, $h_*^\kappa(s)$ etc. are given by (2.10b).

Proof First consider (4.22). From the generating function (4.4) we see that

$$\sum_\mu L_\mu^{a-1}(x;\alpha)C_\mu^{(\alpha)}(z)=\prod_{j=1}^N(1-z_j)\sum_\sigma L_\sigma^a(x;\alpha)C_\sigma^{(\alpha)}(z) \qquad (4.25)$$

Using[30]

$$\prod_{j=1}^N(1-z_j)=\sum_{r=0}^N\frac{(-1)^r}{r!}J_{(1^r)}^{(\alpha)}(z)$$

the Pieri formula[30]

$$J_{1^r}^{(\alpha)}J_\sigma^{(\alpha)}=r!\sum_{\substack{\kappa\\ \kappa/\sigma\text{ a vertical }r\text{-strip}}}\frac{j_\sigma}{j_\kappa}\psi_{\kappa/\sigma}(\alpha)J_\kappa^{(\alpha)}, \qquad (4.26)$$

and the relationship (2.9), the r.h.s. of (4.25) can be rewritten as

$$\sum_{r=0}^N\sum_{\substack{\kappa,\sigma\\ \kappa/\sigma\text{ a vertical }r\text{-strip}}}(-1)^r\alpha^{|\sigma|}|\sigma|!\,L_\sigma^a(x;\alpha)\psi_{\kappa/\sigma}(\alpha)\frac{\alpha^{-|\kappa|}}{|\kappa|!}C_\kappa^{(\alpha)}(z) \qquad (4.27)$$

The result now follows by comparing the coefficients of $C_\kappa^{(\alpha)}(z)$ on the l.h.s. of (4.25) with that in (4.27).

Now consider (4.23). Using the formula [30]

$$\prod_{j=1}^N(1-z_j)^{-1/\alpha}=\sum_{r=0}^\infty\frac{J_{(r)}^{(\alpha)}(z)}{\alpha^r r!}$$

and the Pieri formula (4.26) in conjugate form

$$J_{(r)}^{(\alpha)}(z),J_\sigma^{(\alpha)}(z)=r!\alpha^r\sum_{\substack{\kappa\\ \kappa/\sigma\text{ a horizontal }r\text{-strip}}}\frac{j_\sigma}{j_\kappa}\phi_{\kappa/\sigma}(\alpha)J_\kappa^{(\alpha)}(z),$$

together with the formula (2.9), we have

$$\begin{aligned}\sum_\mu L_\mu^{a+1/\alpha}(x;\alpha)C_\mu^{(\alpha)}(z) &= \prod_{j=1}^N(1-z_j)^{-1/\alpha}\sum_\sigma L_\sigma^a(x;\alpha)C_\sigma^{(\alpha)}(z)\\ &= \sum_{r=0}^\infty\sum_{\substack{\kappa\\ \kappa/\sigma\text{ a horizontal }r\text{-strip}}}L_\sigma^a(x;\alpha)\alpha^{|\sigma|-|\kappa|}\frac{|\sigma|!}{|\kappa|!}\phi_{\kappa/\sigma}(\alpha)C_\sigma^{(\alpha)}(z).\end{aligned}$$

The identity (4.23) follows by equating coefficients of $C_\sigma^{(\alpha)}(z)$.

The analogue of (4.17a) takes the form

Proposition 4.9

$$E_0^{(x)}L_\kappa^a(x;\alpha)=\sum_{r=1}^{\min(N,|\kappa|)}r\alpha^{-r}\sum_{\substack{\sigma\\ \kappa/\sigma\text{ a vertical }r\text{-strip}}}\frac{|\sigma|!}{|\kappa|!}\psi_{\kappa/\sigma}(\alpha)L_\sigma^{a+1}(x;\alpha)$$

where $\psi_{\kappa/\sigma}(\alpha)$ is given by (4.24a).

Proof Applying $E_0^{(x)}$ to the generating function (4.4), we have

$$\begin{aligned} E_0^{(x)} \sum_\kappa L_\kappa^a(x;\alpha) C_\kappa^{(\alpha)}(z) &= \prod_j (1-z_j)^{-a-q} E_0^{(x)} {}_0\mathcal{F}_0\left(-x, \frac{z}{1-z}\right) \\ &= -p_1\left(\frac{z}{1-z}\right) \prod_j (1-z_j)^{-a-q} {}_0\mathcal{F}_0\left(-x, \frac{z}{1-z}\right) \\ &= -p_1\left(\frac{z}{1-z}\right) \prod_j (1-z_j) \sum_\sigma L_\sigma^{a+1}(x;\alpha) C_\sigma^{(\alpha)}(z) \end{aligned} \quad (4.28)$$

Certainly

$$p_1\left(\frac{z}{1-z}\right) \prod_j (1-z_j) = \sum_k z_k \prod_{p\neq k} (1-z_p).$$

If we differentiate w.r.t. t the identity

$$\prod_{i=1}^N (1-z_i t) = \sum_{r=0}^N \frac{(-1)^r}{r!} J_{1^r}^{(\alpha)}(z) t^r$$

giving

$$\sum_{r=1}^N \frac{(-1)^r}{(r-1)!} J_{1^r}^{(\alpha)}(z) t^{r-1} = -\sum_k z_k \prod_{p\neq k} (1-z_p t)$$

and set $t=1$, we obtain

$$-p_1\left(\frac{z}{1-z}\right) \prod_j (1-z_j) = \sum_{r=1}^N \frac{(-1)^r}{(r-1)!} J_{1^r}^{(\alpha)}(z) \quad (4.29)$$

Inserting (4.29) back into (4.28) gives, after some manipulation

$$E_0^{(x)} \sum_\kappa L_\kappa^a(x;\alpha) C_\kappa^{(\alpha)}(z) = \sum_{r=1}^N r\alpha^{-r} \sum_{\substack{\kappa,\sigma \\ \kappa/\sigma \text{ a vertical } r\text{-strip}}} \frac{|\sigma|!}{|\kappa|!} \psi_{\kappa/\sigma}(\alpha) L_\sigma^{a+1}(x;\alpha) C_\kappa^{(\alpha)}(z)$$

which yields the result upon comparison of the coefficients of $C_\kappa^{(\alpha)}(z)$.

4.3 Integration formulas

The classical Laguerre polynomial obeys integration formulas analogous to the integration formulas (3.17) for the classical Hermite polynomial:

$$\int_0^\infty y^a e^{-y} \left(L_k^a(y)\right)^2 dy = \frac{\Gamma(a+1+k)}{k!} \quad (4.30a)$$

$$\frac{k!}{\Gamma(a+1)} e^x \int_0^\infty y^a e^{-y} {}_0F_1(a+1; -xy) L_k^a(y)\, dy = x^k \quad (4.30b)$$

$$\frac{e^x}{k!\Gamma(a+1)} \int_0^\infty y^a e^{-y} {}_0F_1(a+1; -xy) y^k\, dy = L_k(x) \quad (4.30c)$$

The higher-dimensional analogues of these formulas can be established using the generating functions (4.2), (4.4) in much the same way as the higher-dimensional analogues of

(3.17) were established using the generating function (3.2a). To present these results, we will make use of the notation

$$d\mu^{(L)}(y) := \prod_{j=1}^{N} y_j^a \; e^{-y_j} \prod_{1 \le j < k \le N} |y_k - y_j|^{2/\alpha} dy_1 \dots dy_N. \tag{4.31}$$

Proposition 4.10 *We have*

$$\mathcal{N}_\kappa^{(L)} := \int_{[0,\infty)^N} \Big(L_\kappa^a(x;\alpha) \Big)^2 d\mu^{(L)}(x) = \mathcal{N}_0^{(L)} \frac{[a+q]_\kappa^{(\alpha)}}{C_\kappa^{(\alpha)}(1^N)|\kappa|!} \tag{4.32}$$

where

$$\mathcal{N}_0^{(L)} := \int_{[0,\infty)^N} d\mu^{(L)}(x) = \alpha^{(1-N-(N-1)^2/\alpha)} \prod_{j=0}^{N-1} \frac{\Gamma(1+(j+1)/\alpha)\Gamma(a+1+j/\alpha)}{\Gamma(1+1/\alpha)}. \tag{4.33}$$

Proof Multiplication of both sides of the generating function (4.4) by $L_\kappa^a(x;\alpha)$ and integration with respect to $d\mu^{(L)}(x)$ gives, upon using the orthogonality of $\{L_\kappa^a\}_\kappa$ with respect to the inner product (2.21),

$$\prod (1-z)^{-(a+q)} \int_{[0,\infty)^N} {}_0\mathcal{F}_0^{(\alpha)}(-x; \frac{z}{1-z}) L_\kappa^a(x;\alpha)\, d\mu^{(L)}(x) = \mathcal{N}_\kappa^{(L)} C_\kappa^{(\alpha)}(z).$$

Setting $z_1 = \dots = z_N = c$, using (3.16) and changing variables $cx_j/(1-c) =: y_j$ this reads

$$c^{-N(a+q)} \int_{[0,\infty)^N} e^{-p_1(y)/c} \prod_{j=1}^{N} y_j^a \, L_\kappa^a \left(\frac{(1-c)}{c} y_j \right) \prod_{1 \le j < k \le N} |y_k - y_j|^{2/\alpha} \prod_{j=1}^{N} dy_j = \mathcal{N}_\kappa^{(L)} C_\kappa^{(\alpha)}(c^N).$$

The stated result follows by choosing $c = 1$ and noting from (4.14a) that

$$L_\kappa^a(0;\alpha) = [a+q]_\kappa^{(\alpha)} / |\kappa|!. \tag{4.34}$$

The analogues of the formulas (4.30b) and (4.30c) are deduced from the following integration formula.

Proposition 4.11 *We have*

$$\begin{aligned} \int_{[0,\infty)^N} {}_0\mathcal{F}_1^{(\alpha)}(a+q; x; -z_a)\, {}_0\mathcal{F}_1^{(\alpha)}(a+q; x; -z_b)\, d\mu^{(L)}(x) \\ = \mathcal{N}_0^{(L)} e^{-p_1(z_a)} e^{-p_1(z_b)} {}_0\mathcal{F}_1^{(\alpha)}(a+q; z_a; z_b). \end{aligned} \tag{4.35}$$

Proof Substitute for $e^{p_1(z_s)} {}_0\mathcal{F}_1^{(\alpha)}(a+q; x; -z_s)$ $(s = a, b)$ using the generating function (4.2) and integrate with respect to $d\mu^{(L)}(x)$ term-by-term. From the orthogonality property of $\{L_\kappa^a\}_\kappa$ with respect to (2.21), only the diagonal terms in the double sum are non-zero, with the integral then being evaluated according to Proposition 4.10. The resulting sum is identified with ${}_0\mathcal{F}_1^{(\alpha)}$ according to the definition (4.3b).

Corollary 4.1 *We have*

$$\int_{[0,\infty)^N} {}_0\mathcal{F}_1^{(\alpha)}(a+q;x;-z_a) L_\kappa^a(x;\alpha)\, d\mu^{(L)}(x) = \frac{\mathcal{N}_0^{(L)}}{C_\kappa^{(\alpha)}(1^N)|\kappa|!} e^{-p_1(z_a)} C_\kappa^{(\alpha)}(z_a) \tag{4.36a}$$

$$\int_{[0,\infty)^N} {}_0\mathcal{F}_1^{(\alpha)}(a+q;x;-z_a) C_\kappa^{(\alpha)}(x)\, d\mu^{(L)}(x) = \mathcal{N}_0^{(L)} C_\kappa^{(\alpha)}(1^N)|\kappa|! e^{-p_1(z_a)} L_\kappa^a(z_a;\alpha). \tag{4.36b}$$

Proof The integration formula (4.36a) follows from (4.35) after multiplying both sides by $e^{p_1(z_b)}$, using the generating function (4.2) to substitute for $e^{p_1(z_b)}{}_0\mathcal{F}_1^{(\alpha)}(a+q;x;-z_b)$ and equating coefficients of $C_\kappa^{(\alpha)}(z_b)$ on both sides. The integration formula (4.36b) follows from (4.35) after replacing z_b by $-z_b$, substituting for $e^{p_1(z_b)}{}_0\mathcal{F}_1^{(\alpha)}(a+q;z_a;-z_b)$ using (4.2) and equating coefficients of $C_\kappa^{(\alpha)}(z_b)$ on both sides.

Analogous to the Hermite case, the integration formula (4.35) can be used to derive the analogue of the classical summation formula valid for $|t|<1$

$$\sum_{n=0}^\infty \frac{n!}{(a+1)_n} L_n^a(x) L_n^a(y) t^n = (1-t)^{-a-1} \exp\Big(-\frac{t}{1-t}(x+y)\Big) {}_0F_1\left(a+1; \frac{xyt}{(1-t)^2}\right) \tag{4.37}$$

Proposition 4.12 *For $|t|<1$ we have*

$$\begin{aligned} G^{(L)}(x,y;t) &:= \sum_\kappa \frac{L_\kappa(x;\alpha) L_\kappa(y;\alpha)}{\mathcal{N}_\kappa^{(L)}} t^{|\kappa|} \\ &= \frac{1}{\mathcal{N}_0^{(L)}} (1-t)^{-N(a+q)} \exp\Big(-\frac{t}{1-t}(p_1(x)+p_1(y))\Big) {}_0\mathcal{F}_1^{(\alpha)}\left(a+q; \frac{y}{1-t}; \frac{tx}{1-t}\right) \end{aligned}$$

Proof This follows by following the procedure used in the Hermite case, Proposition 3.9.

Notice that in the special case $x=0$ the above summation reduces to elementary functions, giving

$$G^{(L)}(0,y;t) = \frac{1}{\mathcal{N}_0^{(L)}} (1-t)^{-N(a+q)} \exp\Big(-\frac{t}{1-t} p_1(y)\Big). \tag{4.38}$$

Interpretation of this result in terms of an explicit solution of the Fokker-Planck equation (1.6) with W given by (1.3b) will be discussed in the next section.

Finally, we note (prompted by M. Lassalle) that (4.6a) implies the identity

$$\frac{(-1)^{|\kappa|}}{|\kappa|! C_\kappa^{(\alpha)}(1^N)} \exp\Big(-D_1^{(x)} - (a+1)E_0^{(x)}\Big) C_\kappa^{(\alpha)}(x) = L_\kappa^a(x;\alpha) \tag{4.39}$$

(the derivation parallels that of (3.21)). Analogous to (3.22), comparison with (4.36b) gives that for any symmetric analytic function $f(x)$,

$$\frac{e^{p_1(z)}}{\mathcal{N}_0^{(L)}} \int_{[0,\infty)^N} {}_0\mathcal{F}_1^{(\alpha)}(a+q;x;-z) f(-x) d\mu^{(L)}(x) = \exp\Big(-D_1^{(z)} - (a+1)E_0^{(z)}\Big) f(z) \tag{4.40}$$

and hence by a similar argument as before, if

$$F(z) = \frac{e^{p_1(z)}}{\mathcal{N}_0^{(L)}} \int_{[0,\infty)^N} {}_0\mathcal{F}_1^{(\alpha)}(a+q;x;-z) f(-x) d\mu^{(L)}(x) \tag{4.41}$$

then

$$f(z) = \frac{e^{-p_1(z)}}{\mathcal{N}_0^{(L)}} \int_{[0,\infty)^N} {}_0\mathcal{F}_1^{(\alpha)}(a+q;x;z) F(x) d\mu^{(L)}(x) \tag{4.42}$$

(in the case $\alpha = \infty$ (4.41) corresponds to the Hankel transform).

5 Applications

5.1 The ground state global density

As noted in the Introduction, the ground states of the Schrödinger operators (1.2) are, up to normalization, of the form $e^{-\beta W/2}$ where W is given by (1.3). The ground state density in a system of $N+1$ particles, $\rho_{N+1}(x)$ say, is then given by

$$\rho_{N+1}(x) = \frac{N+1}{Z_{N+1}} \prod_{l=1}^{N} \int_I dx_l \, e^{-\beta W(x,x_1,\dots,x_N)} \tag{5.1}$$

where

$$Z_{N+1} := \prod_{l=1}^{N+1} \int_I dx_l \, e^{-\beta W(x_1,x_2,\dots,x_{N+1})} \tag{5.2}$$

An alternative interpretation of (5.1) is as the density at the point x in the statistical mechanical system of $N+1$ particles with potential energy W confined to the interval I, in equilibrium at inverse temperature β.

In the case $W = W^{(H)}$ as given by (1.3a), a physical argument based on the interpretation of the harmonic term as an electrostatic potential (see e.g. [2]) predicts that for all β

$$\lim_{N\to\infty} \sqrt{\frac{2}{N}} \rho(\sqrt{2N}x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2}, & |x|<1 \\ 0, & |x| \ge 1 \end{cases} \tag{5.3}$$

This limit gives the so-called global density, and the result is known as the Wigner semi-circle law.

For $W = W^{(L)}$ the change of variables $y_j = x_j^2$ gives

$$\rho_{N+1}(y) = \frac{N+1}{Z_{N+1}} e^{-\beta y/2} y^{\beta\mu/2} \prod_{l=1}^{N} \int_0^\infty dy_l |y-y_l|^\beta \, e^{-\beta y_l/2} y_l^{\beta\mu/2} \prod_{1\le j<k\le N} |y_k - y_j|^\beta \tag{5.4}$$

where $\mu := a' - 1/\beta$ and

$$Z_{N+1} := \prod_{l=1}^{N+1} \int_0^\infty dy_l \, e^{-\beta y_l/2} y_l^{\beta\mu/2} \prod_{1\le j<k\le N+1} |y_k - y_j|^\beta \tag{5.5}$$

The same type of electrostatics calculation used to obtain (5.3) predicts that for all β and μ,

$$\lim_{N\to\infty} \rho_{N+1}(4Ny) = \begin{cases} \frac{1}{2\pi y^{1/2}}\sqrt{1-y}, & 0<y<1 \\ 0, & y<0, \ y\ge 1 \end{cases} \tag{5.6}$$

Finally, for $W = W^{(J)}$, the change of variable $\sin^2 \phi_j = y_j$ gives

$$\rho_{N+1}(y) = \frac{N+1}{Z_{N+1}} y^{\beta\mu_1/2}(1-y)^{\beta\mu_2/2} \prod_{l=1}^{N} \int_0^1 dy_l \, |y-y_l|^\beta y_l^{\beta\mu_1/2}(1-y_l)^{\beta\mu_2/2} \prod_{1\le j<k\le N} |y_k - y_j|^\beta \tag{5.7}$$

where $\mu_1 := a' - 1/\beta$, $\mu_2 := b' - 1/\beta$ and

$$Z_{N+1} = \prod_{l=1}^{N+1} \int_0^1 dy_l \, y_l^{\beta\mu_1/2}(1-y_l)^{\beta\mu_2/2} \prod_{1\le j<k\le N+1} |y_k - y_j|^\beta \tag{5.8}$$

In this case the electrostatics calculation gives

$$\lim_{N\to\infty} \frac{1}{N}\rho_{N+1}(y) = \begin{cases} \frac{1}{\pi}\frac{1}{\sqrt{y(1-y)}} & 0 < y < 1 \\ 0 & y < 0, \ y > 1 \end{cases} \tag{5.9}$$

In this section we will show how for β even the density (5.1) in the Hermite, Laguerre and Jacobi cases is related to eigenstates of the operator (1.7) and thus the generalized Hermite, Laguerre and Jacobi polynomials respectively (this result is already implicit in earlier publications [9, 10, 17]). Furthermore, we will show how the global density can be evaluated by using integral representations.

5.2 Relationship between the density and the generalized polynomials

Instead of considering the density directly, we proceed as in [9] and introduce a function f depending on the auxilary variables $t_1, \dots, t_m$:

$$f(t_1, \dots, t_m) := \frac{1}{Q_N} \prod_{l=1}^{N} \int_I dy_l \, e^{-\beta V(y_l)} \prod_{s=1}^{m} \prod_{l=1}^{N} (y_l - t_s) \prod_{1\le j<k\le N} |y_k - y_j|^\beta \tag{5.10}$$

where the normalization Q_N is chosen so that f equals unity at the origin. For an appropriate choice of I and V, (5.10) gives each of the densities in the Hermite, Laguerre and Jacobi cases for β even according to the formula

$$\rho_{N+1}(y) = (N+1)\frac{Q_N}{Z_{N+1}} e^{-\beta V(y)} f(t_1, \dots, t_\beta)\Big|_{t_1=\ldots=t_\beta=y} \tag{5.11}$$

Let us consider each case in turn, starting with the Jacobi case.

Jacobi case

Kaneko [17] has shown that with

$$I = [0,1], \quad e^{-\beta V(y)} = y^{\lambda_1}(1-y)^{\lambda_2}, \qquad t := (t_1, \dots, t_m), \quad \lambda_i = \beta\mu_i/2, \quad (i = 1,2) \tag{5.12}$$

$f := f^{(J)}(\lambda_1, \lambda_2, \beta; t)$ as given by (5.10) is the unique solution of each of the p.d.e.'s

$$t_p(1-t_p)\frac{\partial^2 F}{\partial t_p^2} + \Big(c' - \frac{2}{\beta}(m-1) - (a' + b' + 1 - \frac{2}{\beta}(m-1))t_p\Big)\frac{\partial F}{\partial t_p} - a'b'F$$
$$+\frac{2}{\beta}\sum_{\substack{j=1 \\ j\neq p}}^{N} \frac{1}{t_p - t_j}\Big(t_p(1-t_p)\frac{\partial F}{\partial t_p} - t_j(1-t_j)\frac{\partial F}{\partial t_j}\Big) = 0 \tag{5.13}$$

$(p = 1, \dots, m)$ with

$$a' = -N,\ m = \beta,\ b' = \frac{2}{\beta}(\lambda_1 + \lambda_2 + m + 1) + N - 1,\ c' = \frac{2}{\beta}(\lambda_1 + m) \tag{5.14}$$

Furthermore, Kaneko (see also Yan [33]) has shown that the solution of (5.13), normalized to unity at the origin, is also given by the generalized hypergeometric function ${}_2F_1^{(\beta/2)}(a', b'; c'; t)$ where

$${}_pF_r^{(\beta/2)}(a_1, \dots, a_p; b_1, \dots, b_r; t) := \sum_\kappa \frac{1}{|\kappa|!} \frac{[a_1]_\kappa^{(\beta/2)} \cdots [a_p]_\kappa^{(\beta/2)}}{[b_1]_\kappa^{(\beta/2)} \cdots [b_r]_\kappa^{(\beta/2)}} C_\kappa^{(\beta/2)}(t) \tag{5.15}$$

(c.f. (4.3b)), so that

$$f^{(J)}(\lambda_1, \lambda_2, \beta; t) = {}_2F_1^{(\beta/2)}(-N, \frac{2}{\beta}(\lambda_1 + \lambda_2 + m + 1) + N - 1; \frac{2}{\beta}(\lambda_1 + m); t) \tag{5.16}$$

Notice that with $a' = -N$, ${}_2F_1^{(\beta/2)}(a', b'; c'; t)$ as defined by (5.15) indeed terminates and gives a polynomial. To see the connection with the generalized Jacobi polynomials, we note that by summing the p.d.e.'s (5.13) an eigenvalue equation results. The eigenoperator is precisely the operator (2.1c) with

$$N = m,\ a = \frac{2}{\beta}(\lambda_1 + 1) - 1,\ b = \frac{2}{\beta}(\lambda_2 + 1) - 1.$$

Furthermore, from (5.15) and (5.16) $f^{(J)}$ has a Jack polynomial expansion of the form (2.11c) and from the definition (5.10) of f, the highest weight monomial in the power series expansion of f is $m_{(N^m)}$. We thus have

$$f^{(J)}(\lambda_1, \lambda_2, \beta; t) = \tilde{G}_{(N^m)}^{(2(\lambda_1+1)/\beta - 1, 2(\lambda_2+1)/\beta - 1)}(t; \beta/2) \tag{5.17}$$

The tilde here denotes that the normalization in the generalized Jacobi polynomial is such that it equals unity at the origin. Comparison of (5.16) and (5.17) gives an equality between $\tilde{G}$ and ${}_2F_1$ therein.

The formula (5.11) for $\rho_{N+1}(y)$ also requires the value of Q_N and Z_{N+1}. Both quantities are examples of the Selberg integral

$$\begin{aligned} S_N(\lambda_1, \lambda_2, \lambda) &:= \left(\prod_{l=1}^N \int_0^1 dt_l\, t_l^{\lambda_1} (1 - t_l)^{\lambda_2} \right) \prod_{1 \le j < k \le N} |t_k - t_j|^{2\lambda} \\ &= \prod_{j=0}^{N-1} \frac{\Gamma(\lambda_1 + 1 + j\lambda)\Gamma(\lambda_2 + 1 + j\lambda)\Gamma(1 + (j+1)\lambda)}{\Gamma(\lambda_1 + \lambda_2 + 2 + (N + j - 1)\lambda)\Gamma(1 + \lambda)}. \end{aligned} \tag{5.18}$$

We have

$$Z_{N+1} = S_{N+1}(\beta\mu_1/2, \beta\mu_2/2, \beta/2), \quad Q_N = S_N(\beta\mu_1/2 + \beta N, \beta\mu_2/2, \beta/2). \tag{5.19}$$

Substituting (5.17) and (5.19) in (5.11) gives

$$\begin{aligned} \rho_{N+1}(y) = (N+1) &\frac{S_N(\beta\mu_1/2 + \beta N, \beta\mu_2/2, \beta/2)}{S_{N+1}(\beta\mu_1/2, \beta\mu_2/2, \beta/2)} y^{\beta\mu_1/2} (1 - y)^{\beta\mu_2/2} \\ &\times \tilde{G}_{(N^\beta)}^{(2/\beta + \mu_1 - 1, 2/\beta + \mu_2 - 1)}(t_1, \dots, t_\beta; \beta/2)\Big|_{t_1 = \dots = t_\beta = y} \end{aligned} \tag{5.20}$$

Our ability to compute the global limit relies on an integral representation of ${}_2F_1^{(1/\lambda)}$ (different of course from (5.10)) and thus, after equating (5.16) and (5.17), of $\tilde{G}_{(N^m)}$. This integral representation can be derived from the integral representation of the generalized hypergeometric function [33]

$$\begin{aligned} &{}_2F_1^{(1/\lambda)}(a, \lambda(m-1)+\nu_1+1; 2\lambda(m-1)+\nu_1+\nu_2+2; t) \\ &\quad = \frac{1}{S_m(\nu_1,\nu_2,\lambda)} \int_{[0,1]^m} dx_1 \dots dx_m \, {}_1\mathcal{F}_0^{(1/\lambda)}(a;t;x) D_{\nu_1,\nu_2,\lambda}(x) \end{aligned} \tag{5.21}$$

with ${}_1\mathcal{F}_0^{(1/\lambda)}$ given by (4.3b), and

$$D_{\nu_1,\nu_2,\lambda}(x) := \prod_{j=1}^{m} x_j^{\nu_1}(1-x_j)^{\nu_2} \prod_{1 \le j < k \le m} |x_k - x_j|^{2\lambda} \tag{5.22}$$

Since $\tilde{G}$ in (5.20) is equal to ${}_2F_1^{(\beta/2)}$ in (5.16) with $m=\beta$, we must set

$$a = -N, \ \lambda = 2/\beta, \ \nu_1 = \frac{4}{\beta} + \mu_1 + \mu_2 + N - 2, \ \nu_2 = -2 - \mu_2 - N \tag{5.23}$$

in (5.21).

We note that ν_2 is negative so that (5.21) is not defined as written. However we can readily analytically continue the integral (5.21) so that it is valid for ν_2 negative by following the procedure detailed in [10]. Thus we deform the contours $[0,1]^m$ to the contours $\mathcal{C}^m$, where $\mathcal{C}$ is any simple closed contour which starts at the origin and encircles the point $x=1$ (this is first done under the assumption that ν_2 is not an integer, and λ is an integer; it is extended to all ν_2 by analytic continuation and to all λ by noting that the r.h.s. is analytic in λ when it is defined, while the l.h.s. is a rational function of λ in the case of interest ($a=-N$)). Furthermore, we have the formula[17]

$${}_1\mathcal{F}_0^{(1/\lambda)}(a;t;x)\Big|_{t_1,\dots,t_\beta = c} = \prod_{l=1}^{m}(1-cx_l)^{-a} \tag{5.24}$$

Thus we have

$$\begin{aligned} &\tilde{G}_{(N^\beta)}^{(2/\beta+\mu_1-1, 2/\beta+\mu_2-1)}(t_1,\dots,t_\beta;\beta/2)\Big|_{t_1=\dots=t_\beta=y} = \\ &\frac{1}{C}\int_{\mathcal{C}^\beta} dx_1 \dots dx_\beta \prod_{l=1}^{\beta}(1-yx_l)^N x_l^{4/\beta+\mu_1+\mu_2+N-2}(1-x_l)^{-2-\mu_2-N} \prod_{1\le j<k\le\beta}|x_k-x_j|^{4/\beta} \end{aligned} \tag{5.25}$$

where C is chosen so that at $t=0$, $\tilde{G}$ is unity. This is our desired integral representation.

Laguerre case

Let (5.10) with Laguerre weight $e^{-\beta V(y)} = e^{-\beta y/2} y^{\beta\mu/2}$ and integration interval $I=[0,\infty)$ be denoted $f = f^{(L)}(\mu,\beta;t)$. Comparison with the definition of $f^{(J)}$ shows

$$\lim_{L\to\infty}\left(\frac{1}{L}\right)^{mN} f^{(J)}(\beta\mu/2, \beta L/2, \beta; t/L) = f^{(L)}(\beta\mu/2, \beta; t) \tag{5.26}$$

Substituting (5.16) and (5.17) for $f^{(J)}$ and using (2.19) and the fact that $\lim_{b\to\infty} {}_2F_1^{(\alpha)}(a,b;c;x/b) = {}_1F_1^{(\alpha)}(a;c;x)$, we thus have

$$f^{(L)}(\beta\mu/2;t) = \tilde{L}_{(N^m)}^{\mu-1+2/\beta}(t;\beta/2) = {}_1F_1^{(\beta/2)}(-N;\mu+2;t) \tag{5.27}$$

where $\tilde{L}$ denotes the generalized Laguerre polynomial normalized to unity at the origin. (The equality between $f^{(L)}$ and ${}_1F_1^{(\beta/2)}$ has previously been given in [10] and the equality between $\tilde{L}_{(N^m)}^{\mu-1+2/\beta}$ and ${}_1F_1^{(\beta/2)}$ has been noted in [23].) Furthermore from the working in [10] we have

$$
{}_1F_1^{(\beta/2)}(-N;\mu+2;t_1,\dots,t_\beta)\Big|_{t_1=\dots t_\beta=y} =
$$
$$
\frac{1}{C'}\int_{(\mathcal{C}')^\beta} dx_1\cdots dx_\beta \prod_{j=1}^{\beta} e^{yx_j} x_j^{-N-3+2/\beta}(1-x_j)^{\mu+N+2/\beta-1} \prod_{1\le j<k\le\beta} |x_k-x_j|^{4/\beta} \tag{5.28}
$$

Hermite case

Starting with I and $e^{-\beta V(y)}$ given by (5.12), the Hermite case $I=(-\infty,\infty)$ and $e^{-\beta V(y)} = e^{-\beta y^2/2}$ can be obtained by the change of variables and limiting procedure

$$
y_j \mapsto \frac{1}{2}(1-\frac{y_j}{L}), \quad t_j \mapsto \frac{1}{2}(1-\frac{t_j}{L}), \quad \beta\mu_1/2 = \beta\mu_2/2 = \beta L^2/2, \quad L\to\infty \tag{5.29}
$$

Hence from (2.18) and (5.17) with $\lambda_1=\lambda_2=\beta L^2/2$ we see that in the Hermite case, $f^{(H)}$ is proportional to $H_{(N^m)}(t_1,\dots,t_m;\beta/2)$ (the fact that $f^{(H)}$ is an eigenfunction of (2.1a) with eigenvalue $-2N$ was shown in [9]). Thus, from (5.1), if we denote by $\bar{H}_\kappa$ the generalized Hermite polynomial normalized so that the coefficient of the highest weight monomial m_κ is unity, we have

$$
\rho_{N+1}(x) = (N+1)\frac{Z_N}{Z_{N+1}} e^{-\beta x^2/2}\bar{H}_{(N^\beta)}(t_1,\dots,t_\beta;\beta/2)\Big|_{t_1=\dots t_\beta=x} \tag{5.30a}
$$

where

$$
\begin{aligned}
Z_N &= \prod_{l=1}^{N}\int_{-\infty}^{\infty} d\lambda_l \,\exp\left(-\frac{\beta}{2}\sum_{j=1}^{N}\lambda_j^2\right)\prod_{1\le j<k\le N}|\lambda_k-\lambda_j|^\beta \\
&= \beta^{-N/2-N\beta(N-1)/4}(2\pi)^{N/2}\prod_{j=0}^{N-1}\frac{\Gamma(1+\beta(j+1)/2)}{\Gamma(1+\beta/2)}
\end{aligned} \tag{5.30b}
$$

(compare (5.30b) with $\mathcal{N}_0^{(H)}$ in Proposition 3.7). To obtain a form of $\bar{H}_{(N^\beta)}$ suitable for asymptotic analysis, we make use of the integral representation Corollary 3.2 of H_κ. In the case of interest ($\kappa=(N^\beta)$, $t_1=\dots=t_\beta=x$) we have

$$
{}_0\mathcal{F}_0^{(2/\beta)}(2y_1,\dots,2y_\beta;-iz_1,\dots,-iz_\beta)\Big|_{z_1=\dots=z_\beta=x} = \prod_{j=1}^{\beta} e^{-2ixy_j}
$$

and

$$
C_{(N^\beta)}^{(2/\beta)}(iy_1,\dots,iy_\beta) = \prod_{j=1}^{\beta}(iy_j)^N
$$

so we can complete the square in the integrand of the formula of Corollary 3.2 and change variables to obtain

$$
\bar{H}_{(N^\beta)}(t_1,\dots,t_\beta;\beta/2)\Big|_{t_1=\dots=t_\beta=x} = \frac{1}{V_\beta}\int_{\mathbb{R}^\beta} du_1\dots du_\beta \prod_{j=1}^{\beta}(iu_j+x)^N e^{-u_j^2}\prod_{1\le j<k\le\beta}|u_k-u_j|^{4/\beta} \tag{5.31}
$$

where

$$V_m := \mathcal{N}_0^{(H)}(N=m, \alpha=\beta/2) \tag{5.32}$$

(It is also possible to derive (5.31) by performing the limiting procedure (5.29) in the integral representation (5.25).)

Analogous to the situation in (5.25) and (5.28), we note that each integration path along the real line can be deformed to the path $\mathcal{C}''$, where $\mathcal{C}''$ is a simple contour which starts at $-\infty$ and ends at ∞ (this is true for $2/\beta \in \mathbb{Z}_{\geq 0}$ by Cauchy's theorem; it then follows for all values of $2/\beta$ that the r.h.s. is defined by noting that the r.h.s. is then analytic in $2/\beta$ while the l.h.s. is a rational function in this variable).

5.3 The global density limit

Using the integral representations (5.25), (5.28) and (5.31), the global density limits in (5.3), (5.6) and (5.9) can be computed for all β even. The method used in each case is to deform the contours so that they pass through the saddle points (for each integration variable there are two saddle points), and to expand the integrand in the neighbourhood of these points. Due to the similarities of the three calculations, we will give the details in the Hermite case only.

Changing variables $u_l \mapsto \sqrt{2N} u_l$ in the integral representation (5.31), and substituting the result in (5.30a) gives

$$\begin{aligned} \rho_{N+1}(\sqrt{2N}x) &= (N+1)\frac{Z_N}{Z_{N+1}V_\beta}(2N)^{(\beta N+3\beta-2)/2}e^{-\beta N x^2} \\ &\quad \times \int_{\mathbb{R}^\beta} du_1 \dots du_\beta \prod_{l=1}^{\beta} e^{-2Nu_l^2}(iu_l+x)^N \prod_{1\leq j<k\leq\beta} |u_k-u_j|^{4/\beta} \end{aligned} \tag{5.33}$$

For each integration variable u_l the N-dependent terms in the integrand are

$$e^{-2Nu_l^2}(iu_l+x)^N = e^{-2Nu_l^2+N\log(iu_l+x)}$$

The exponent has a stationary point when

$$u_l = u_\pm := \frac{ix}{2} \pm \frac{1}{2}(1-x^2)^{1/2}, \tag{5.34}$$

so according to the saddle point method of asymptotic analysis we should deform each of the contours of integration in (5.33) to pass through u_+ and u_-.

With the contours of integration so deformed, we must expand the integrand in the neighbourhood of the saddle points. Due to the factor $\prod_{1\leq j<k\leq\beta}|u_k-u_j|^{4/\beta}$ the maximum contribution will be obtained by expanding $\beta/2$ integration variables ($u_1,\dots,u_{\beta/2}$ say) about u_+ and the remaining $\beta/2$ integration variables ($u_{\beta/2+1},\dots,u_\beta$) about u_-. This specific choice is only one of the $\binom{\beta}{\beta/2}$ equivalent ways of dividing the integration variables into these two classes, so after expanding the variables with the specific choice we must multiply by the combinatorial factor.

In the neighbourhood of the specified points we have

$$\begin{aligned} & e^{-2Nu_l^2+N\log(iu_l+x)} \\ &\quad \sim \exp[-2Nu_\pm^2 + N\log(iu_\pm+x) - \frac{1}{2}(u-u_\pm)^2(4N-\frac{N}{(iu_\pm+x)^2})] \end{aligned} \tag{5.35}$$

where on the r.h.s. u_+ is to be taken for $j = 1, \dots, \beta/2$ while u_- is to be taken for $j = \beta/2 + 1, \dots, \beta$. Also

$$\prod_{1 \le j < k \le \beta} |u_k - u_j|^{4/\beta} \sim |u_+ - u_-|^\beta \prod_{1 \le j < k \le \beta/2} |u_k - u_j|^{4/\beta} \prod_{\beta/2+1 \le j < k \le \beta} |u_k - u_j|^{4/\beta} \quad (5.36)$$

Thus after substituting (5.35) and (5.36) in (5.33) we obtain

$$\rho_{N+1}(\sqrt{2N}x) \sim$$

$$N \frac{Z_N}{Z_{N+1} V_\beta} (2N)^{(\beta N + 3\beta - 2)/2} e^{-\beta N x^2} e^{-N\beta(u_+^2 + u_-^2) + (N\beta/2)\log|iu_+ + x|^2} |u_+ - u_-|^\beta \begin{pmatrix} \beta \\ \beta/2 \end{pmatrix}$$

$$\times \left| \int_{\mathbb{R}^{\beta/2}} du_1 \dots du_{\beta/2} \prod_{l=1}^{\beta/2} \exp[-2Nu_l^2(2N - \frac{N}{2(iu_+ + x)^2})] \prod_{1 \le j < k \le \beta/2} |u_k - u_j|^{4/\beta} \right|^2 \quad (5.37)$$

To simplify (5.37) note that a simple change of variables gives that the last line is equal to

$$\left(\frac{1}{|2N - \frac{N}{2(iu_+ + x)^2}|} \right)^{\beta - 1} (V_{\beta/2})^2,$$

where $V_{\beta/2}$ is defined by (5.32). Now suppose $x < 1$ so that $u_-^* = -u_+$. Using (5.34) we then have

$$\left| 2N - \frac{N}{2(iu_+ + x)^2} \right| = 4N(1 - x^2)^{1/2}, \quad |u_+ - u_-| = (1 - x^2)^{1/2},$$

$$u_+^2 + u_-^2 = \frac{1}{2} - x^2, \quad |iu_+ + x| = \frac{1}{2}.$$

Making these substitutions in (5.37) shows that

$$\rho_{N+1}(\sqrt{2N}x) \sim (1 - x^2)^{1/2} N \frac{Z_N}{Z_{N+1}} \frac{(V_{\beta/2})^2}{V_\beta} \begin{pmatrix} \beta \\ \beta/2 \end{pmatrix} e^{-N\beta/2} 2^{-N\beta/2 - \beta/2 + 1} N^{\beta(N+1)/2} \quad (5.38)$$

To simplify the x-independent terms in (5.38) we note from the specific formula (5.30b) and Stirling's formula that

$$\begin{aligned} \frac{Z_N}{Z_{N+1}} &= \beta^{(1+\beta N)/2} (2\pi)^{-1/2} \frac{\Gamma(1 + \beta/2)}{\Gamma(1 + (N+1)\beta/2)} \\ &\sim \frac{\Gamma(\beta/2 + 1)}{\pi} 2^{\beta N/2 - 1/2} N^{-(\beta/2)(N+1) - 1/2} (\beta/2)^{-(\beta/2)} e^{N\beta/2} \end{aligned} \quad (5.39)$$

Also, from (5.32) and straightforward manipulation of the explicit formula in Proposition 3.7 we have

$$\frac{(V_{\beta/2})^2}{V_\beta} = 2^{\beta/2} (\beta/2)^{\beta/2} \frac{\Gamma(1 + \beta/2)}{\Gamma(1 + \beta)} \quad (5.40)$$

Substituting (5.39) and (5.40) in (5.38) gives

$$\rho_{N+1}(\sqrt{2N}x) \sim \frac{\sqrt{2N}}{\pi} (1 - x^2)^{1/2}, \qquad |x| \le 1 \quad (5.41)$$

which is precisely the formula (5.3) for $|x| \le 1$.

For the intervals $|x| > 1$, instead of repeating the working of the expansion about the saddle points (which are both pure imaginary in this case), we note from the result (5.41) that

$$\int_{-1}^{1} \rho(\sqrt{2N}x)\, d(\sqrt{2N}x) \ \sim\ \frac{2N}{\pi}\int_{-1}^{1}(1-x^2)^{1/2}dx = N$$

But from the definition of the density it is non-negative and satisfies the normalization

$$\int_{-\infty}^{\infty} \rho(\sqrt{2N}x)d(\sqrt{2N}x) = N.$$

Thus we must have

$$\frac{\rho(\sqrt{2N}x)}{\sqrt{2N}} \to 0, \quad \text{for} \quad |x| > 1,$$

as predicted by (5.3).

5.4 Initial value problems

The summations $G^{(H)}(w,z;t)$ in Proposition 3.9 and $G^{(L)}(x,y;t)$ in Proposition 4.12 are essentially the Green functions for the solution of the Fokker-Planck equation (1.6) with W given by (1.3a) and (1.3b) respectively. To see this, we first recall that $P = \tilde{G}(x^{(0)}|x;\tau)$ is the Green function solution of (1.6) if it is the solution which satisfies the initial condition

$$P(x;\tau)\Big|_{\tau=0} = \prod_{l=1}^{N} \delta(x_l - x_l^{(0)}), \qquad\qquad x_1^{(0)} < \cdots < x_N^{(0)}$$

By appluing the transformation (1.4) the Fokker-Planck equation can be written as the Schrödinger equation (1.5), where $t = \tau/i\beta$. In general the Green function solution of the Schrödinger equation, $G(x^{(0)}|x;t)$ say, may be written in terms of the eigenvalues and eigenfunctions of H. Thus suppose $\{\psi_\kappa\}_\kappa$ is a complete set of orthogonal eigenfunctions of H with corresponding eigenvalues $\{E_\kappa\}_\kappa$. Then the method of separation of variables gives

$$G(x^{(0)}|x;t) = \sum_\kappa \frac{\psi_\kappa(x^{(0)})\psi_\kappa(x)}{\mathcal{N}_\kappa}\, e^{-itE_\kappa}$$

Thus for the Fokker-Planck equation

$$\tilde{G}(x^{(0)}|x;\tau) = e^{\tau E_0/\beta}\frac{\psi_0(x)}{\psi_0(x^{(0)})}G(x^{(0)}|x;\tau/i\beta) \tag{5.42}$$

Now, for the Schrödinger operators (1.2a), (1.2b)

$$\psi_\kappa^{(H)}(x) = \psi_0^{(H)}(x)H_\kappa(x/\sqrt{\alpha};\alpha) \qquad E_\kappa^{(H)} = E_0^{(H)} + \frac{2}{\alpha}|\kappa| \tag{5.43a}$$

$$\psi_\kappa^{(L)}(x) = \psi_0^{(L)}(x)L_\kappa^{(a'/\alpha-1/2)}(x^2/\alpha;\alpha) \qquad E_\kappa^{(L)} = E_0^{(L)} + \frac{4}{\alpha}|\kappa| \tag{5.43b}$$

where

$$\psi_0^{(H)}(x) := \prod_{j=1}^{N} e^{-x_j^2/2\alpha} \prod_{1\le j<k\le N} |x_k - x_j|^{1/\alpha}$$

$$\psi_0^{(L)}(x) := \prod_{j=1}^{N} x_j^{a'/\alpha}\, e^{-x_j^2/2\alpha} \prod_{1\le j<k\le N} |x_k^2 - x_j^2|^{1/\alpha}$$

Substituting (5.43a) and (5.43b) into (5.42) and comparing with the definitions of $G^{(H)}(w,z;t)$ and $G^{(L)}(x,y;t)$ shows that

$$\tilde{G}^{(H)}(x^{(0)}|x;\tau) = \alpha^{-Nq/2}\left(\psi_0^{(H)}(x)\right)^2 G^{(H)}(x^{(0)}/\sqrt{\alpha}, x/\sqrt{\alpha}; e^{-\tau}) \tag{5.44a}$$

$$\tilde{G}^{(L)}(x^{(0)}|x;\tau) = \alpha^{-N(a'/\alpha-1/2+q)}\left(\psi_0^{(L)}(x)\right)^2 G^{(L)}((x^{(0)})^2/\alpha, x^2/\alpha; e^{-2\tau})\Big|_{a=(a'/\alpha-1/2)} \tag{5.44b}$$

From (3.20) and (4.38) we see from (5.44) that for some initial conditions it is possible to express $\tilde{G}^{(H)}$ and $\tilde{G}^{(L)}$ in terms of elementary functions. Thus for $x^{(0)}=c$ (i.e. $x_1^{(0)}=\cdots=x_N^{(0)}=c$) in the Hermite case, from (3.20) and (5.44a) we have

$$\begin{aligned}\tilde{G}^{(H)}(x^{(0)}|x;\tau)\Big|_{x^{(0)}=c} &= \frac{1}{\mathcal{N}_0^{(H)}}\left(\alpha(1-e^{-2\tau})\right)^{-Nq/2} \\ &\times \exp\left(-\frac{1}{\alpha(1-e^{-2\tau})}\sum_{j=1}^N (x_j - e^{-\tau}c)^2\right)\prod_{1\le j<k\le N}|x_k-x_j|^{2/\alpha}\end{aligned} \tag{5.45a}$$

while for $x^{(0)}=0$ in the Laguerre case, (4.38) and (5.44b) give

$$\begin{aligned}\tilde{G}^{(L)}(x^{(0)}|x;\tau)\Big|_{x^{(0)}=0} &= \frac{1}{\mathcal{N}_0^{(L)}}\Big|_{a=a'/\alpha-1/2}\left(\alpha(1-e^{-2\tau})\right)^{-N(a'/\alpha-1/2+q)} \\ &\times \prod_{j=1}^N x_j^{2a'/\alpha}\exp\left(-\frac{1}{\alpha(1-e^{-2\tau})}\sum_{j=1}^N x_j^2\right)\prod_{1\le j<k\le N}|x_k^2-x_j^2|^{2/\alpha}\end{aligned} \tag{5.45b}$$

Now that these explicit solutions have been revealed, they can be verified independent of the theory of generalized classical polynomials, by direct substitution into (1.6) with the appropriate W.

Another consequence of (5.44) is that it implies the asymptotic small-τ behaviour of

$${}_0\mathcal{F}_0^{(\alpha)}\Big(\frac{x^{(0)}}{\tau^{1/2}};\frac{x}{\tau^{1/2}}\Big) \quad \text{and} \quad {}_0\mathcal{F}_1^{(\alpha)}\Big(a+q;\frac{(x^{(0)})^2}{2\tau};\frac{x^2}{2\tau}\Big).$$

Thus in general, as $\tau\to 0$ the asymptotic solution of the Schrödinger equation (1.5) (with $t=\tau/i\beta$) is given by

$$G(x^{(0)}|x;\tau/i\beta) \sim \Big(\frac{\beta}{4\pi\tau}\Big)^{N/2}\prod_{j=1}^N e^{-\beta(x_j-x_j^{(0)})^2/4\tau}$$

Substituting this in (5.42), substituting the result in (5.44) and using Propositions 3.9 and 4.12 gives

$${}_0\mathcal{F}_0^{(\alpha)}\left(\frac{x^{(0)}}{\tau^{1/2}};\frac{x}{\tau^{1/2}}\right) \sim \frac{\pi^{-N/2}2^{N(N-1)/2\alpha}\mathcal{N}_0^{(H)}}{\left(\prod_{1\le j<k\le N}(x_j-x_k)(x_j^{(0)}-x_k^{(0)})/\tau\right)^{1/\alpha}}\prod_{j=1}^N e^{x_jx_j^{(0)}/\tau} \tag{5.46}$$

and

$$\begin{aligned}{}_0\mathcal{F}_1^{(\alpha)}\left(a+q;\frac{(x^{(0)})^2}{2\tau};\frac{x^2}{2\tau}\right) &\sim \frac{\pi^{-N/2}2^{N(a+1/2)+N(N-1)/\alpha}\mathcal{N}_0^{(L)}}{\left(\prod_{1\le j<k\le N}(x_j^2-x_k^2)((x_j^{(0)})^2-(x_k^{(0)})^2)/\tau\right)^{1/\alpha}} \\ &\times \prod_{j=1}^N (x_jx_j^{(0)}/\tau)^{-(a+1/2)}\prod_{j=1}^N e^{x_jx_j^{(0)}/\tau}\end{aligned} \tag{5.47}$$

where it is assumed $x_1 < \cdots < x_N$ and $x_1^{(0)} < \cdots < x_N^{(0)}$. In the case $\alpha = 2$ these asymptotic formulas are known in the mathematical statistics literature (see e.g. [26])

6 A brief literature survey

To our knowledge, the generalized classical polynomials were first introduced by Herz [12] in the case $\alpha = 2$ via integral formulas over measures associated with spaces of orthogonal matrices (however it should be noted that what Herz calls generalized Hermite polynomials do not correspond to the generalized Hermite polynomial we have considered). Constantine and Muirhead extended the work of Herz on the generalized Laguerre polynomials in the case $\alpha = 2$, and derived the formulas (4.2) [27, ex. 7.19], (4.4) [27, Thm. 7.6.3], (4.12) [27, ex. 7.20], (4.14a) [27, eq. 7.6(4)], (4.32) [27, Thm. 7.6.5], (4.36) [27, Thm. 7.6.4] and Proposition 4.12 [27, ex. 7.21] in that case (in comparing formulas it should be noted that Muirhead adopts the normalization $c_{\kappa\kappa} = (-1)^{|\kappa|}/C_\kappa^{(\alpha)}(1^N)$ which is $|\kappa|!$ times the normalization we have used). For general α and $N = 2$ Yan [34] derived (4.4) [34, eq. (5.6)] (4.14a) [34, last eqn. p. 251], (4.32) [34, eq. (5.11)] and (4.36b) [34, eq. (5.13)] (Yan uses the same normalization as Muirhead). For general α Lassalle [19] has reported the results (4.14) and (4.32), and simultaneous to our investigations has obtained the results (4.2) and (4.39) - (4.42) (Lassalle uses the normalization $L_\kappa^a(0;\alpha) = 1$). For values of α corresponding to Jordan algebras, generalized Laguerre polynomials have been studied by Dib [5].

In an unpublished handwritten manuscript Macdonald [24] has derived some properties of the generalized classical polynomials. His results for the generalized Laguerre polynomials, which overlap with the same equations of ours as does the work of Yan, are typically proved for $\alpha = 1/2, 1$ and 2, and are conjectured to remain valid for general α. The validity of a number of the results in [24] for general α rely on a conjecture for the so called generalized Laplace transform of the Jack polynomial:

$$\int_{[0,\infty)^N} {}_0\mathcal{F}_0^{(\alpha)}(-x;y) C_\kappa^{(\alpha)}(x) \prod_{j=1}^N x_j^a \prod_{1\le j<k\le N} |x_k - x_j|^{2/\alpha}\, dx_1 \dots dx_N$$
$$= [a+q]_\kappa^{(\alpha)} C_\kappa^{(\alpha)}(1^N) \prod_{j=1}^N y_l^{-(a+q)} C_\kappa^{(\alpha)}(\frac{1}{y}) \tag{6.1}$$

This conjecture can be proved using results contained herein. First we calculate the generalized Laplace transform of the generalized Laguerre polynomial:

$$\int_{[0,\infty)^N} {}_0\mathcal{F}_0^{(\alpha)}(-x;y) L_\sigma^a(x;\alpha) \prod_{j=1}^N x_j^a \prod_{1\le j<k\le N} |x_k - x_j|^{2/\alpha} dx_1 \dots dx_N$$
$$= \mathcal{N}_\sigma^{(L)} \prod_{j=1}^N y_l^{-(a+q)} C_\sigma^{(\alpha)}(1 - \frac{1}{y}), \tag{6.2}$$

which follows from the first equation of the proof of Proposition 4.10 after noting from (4.15) that

$$\prod_{j=1}^N e^{-x_j}\, {}_0\mathcal{F}_0^{(\alpha)}(-x; \frac{z}{1-z}) = {}_0\mathcal{F}_0^{(\alpha)}(-x; \frac{1}{1-z})$$

and writing $1/(1-z) =: y$. We now use (4.14b) and multiply (6.1) by a suitable σ–dependent factor so that after summing over σ we can replace $L^a_\sigma(x;\alpha)$ on the l.h.s. by $C^{(\alpha)}_\kappa(x)$. On the r.h.s. we then have

$$\prod_{j=1}^N y_l^{-(a+q)}[a+q]^{(\alpha)}_\kappa C^{(\alpha)}_\kappa(1^N) \sum_{\sigma\subseteq\kappa} \binom{\kappa}{\sigma} \frac{(-1)^{|\sigma|}\mathcal{N}^{(L)}_\sigma C_\sigma(1-\frac{1}{y})}{[a+q]^{(\alpha)}_\sigma} \tag{6.3}$$

Substituting the value of $\mathcal{N}^{(L)}_\sigma$ from (4.32) and using (2.14) to compute the sum gives (6.1) as required.

The generalized Hermite polynomials of the type considered in this paper appear to have been first considered by James [14] in the case $\alpha = 2$. Subsequently, for general α Lassalle [21] noted the orthogonality with respect to the measure (2.20), the normalization of Proposition 3.7 and the property of Proposition 3.3. Furthermore, in handwritten notes Lassalle [19] has established Proposition 3.1 and has stated Corollaries 3.1 and 3.2, (3.21), (3.22) and (3.25). Also given in the notes is an explicit formula for the coefficients $c^{(H)}_{\mu\kappa}$ in (2.11a). Macdonald [24] has also considered properties of the generalized Hermite polynomials in the form of conjectures based on derivations in the cases $\alpha = 1/2$, 1 and 2. He has obtained the normalization of Proposition 3.7, the property of Proposition 3.3 and the integration formula of Proposition 3.8 and the generating function of Proposition 3.1.

M. Lassalle has pointed out to us that the exponential operator formulas (3.21) and (4.39) imply an intimate connection between the theory of generalized Hermite and Laguerre polynomials and theory developed by Dunkl [6, 7]. Inspection of these works show that this is indeed so. The Hermite case is the most straightforward, which in the language of [6, 7] corresponds to the root system A_N. Dunkl introduces the operators

$$T_i := \frac{\partial}{\partial x_i} + \frac{1}{\alpha}\sum_{\substack{j=1\\ j\neq i}}^N \frac{1-M_{ij}}{x_i - x_j}$$

where M_{ij} is the operator which exchanges coordinates x_i and x_j. When acting on functions symmetric in $x_1,\dots,x_N$ these operators are related to D_0 (recall (2.12)) by

$$D_0 = \sum_{i=1}^N T_i^2.$$

Also introduced is the pairing $[p,q]_H$. For polynomials p and q homogeneous of the same degree

$$[p,q]_H := p(T^x)q(x), \tag{6.4}$$

where $p(T^x)$ means that each variable x_i in p is replaced by T_i (the ordering within the monomials does not matter since the operators $\{T_i\}$ commute), while if the degrees differ

$$[p,q]_H := 0.$$

This pairing is intimately related to the exponential operator in (3.21). Thus, as noted in [19], it follows from [6, Thm. 3.10] that for homogeneous symmetric polynomials p and q

$$[p,q]_H = \frac{1}{\mathcal{N}^{(H)}_0}\int_{(-\infty,\infty)^N} \left(e^{-D_0/4}p\right)\left(e^{-D_0/4}q\right)\, d\mu^{(H)}(x).$$

From (3.21) and the orthogonality of $\{H_\kappa\}$ with respect to the inner product (2.20) we immediately have the result that the Jack polynomials are orthogonal with respect to the pairing (6.4):

$$[J_\kappa^{(\alpha)}, J_\mu^{(\alpha)}]_H = (2\alpha)^{-|\kappa|} j_\kappa J_\kappa^{(\alpha)}(1^N)\delta_{\kappa,\mu}$$

where we have used Proposition 3.7 and (2.9)

Dunkl also introduces a kernel $K(x,y)$, which for the root system A_N and p a symmetric homogeneous polynomial has the property [7, Prop. 2.1]

$$p(y) = \frac{e^{-p_2(y)}}{\mathcal{N}_0^{(H)}} \int_{(-\infty,\infty)^N} \left(e^{-D_0/4} p\right) K(x,y)\, d\mu^{(H)}(x)$$

(we have changed variables $x \mapsto \sqrt{2}x$, $y \mapsto \sqrt{2}y$ and replaced $K(\sqrt{2}x, \sqrt{2}y)$ by $K(x,y)$). Comparison with Corollary 3.1 (after substituting for H_κ using (3.21)) gives the explicit formula

$$K(x,y) = {}_0\mathcal{F}_0^{(\alpha)}(2x;y).$$

In the Laguerre case there are analogous connections with the work of Dunkl, with the underlying root system now being B_N. The operators T_i are now given by (see e.g. [13])

$$T_i = \frac{\partial}{\partial x_i} + \frac{1}{\alpha} \sum_{\substack{j=1 \\ j\neq i}}^{N} \Big(\frac{1 - M_{ij}}{x_i - x_j} + \frac{1 - S_i S_j M_{ij}}{x_i + x_j} \Big) + \frac{a + 1/2}{x_i}(1 - S_i) \tag{6.5}$$

where the action of the operator S_i is to replace the variable x_i by $-x_i$. When acting on a function f symmetric and even in $x_1, \dots, x_N$ these operators are such that

$$\sum_{i=1}^{N} T_i^2 \Big|_{x_i^2 = u_i} = 4\Big(D_1^{(u)} + (a+1)E_0^{(u)}\Big).$$

Using (6.5) and setting $P(x) := p(x^2)$ and $Q(x) = q(x^2)$, [6, Thm. 3.10] gives

$$[P,Q]_L = \frac{1}{\mathcal{N}_0^{(L)}} \int_{[0,\infty)^N} \left(e^{-(D_1 + (a+1)E_0)} p\right) \left(e^{-(D_1 + (a+1)E_0)} q\right) d\mu^{(L)}(x),$$

where $[\,,\,]_L$ is defined by (6.4) with T_i specified by (6.5). Orthogonality of $\{L_\kappa\}$ with respect to (2.21) and use of (4.39) then gives

$$[J_\kappa^{(\alpha)}, J_\mu^{(\alpha)}]_L = \alpha^{-|\kappa|} j_\kappa [a+q]_\kappa^{(\alpha)} J_\kappa^{(\alpha)}(1^N)\delta_{\kappa,\mu}.$$

Also, (4.36a) with the substitution of (4.39) and the change of variables $x, z_a \mapsto x^2, z_a^2$ gives the kernel $K(x,,y)$ in the B_N case as ${}_0\mathcal{F}_1^{(\alpha)}(a+q; x^2; -y^2)$.

In the context of the Calogero-Sutherland model the expansion of the generalized Hermite polynomials in terms of monomial symmetric functions has been considered by Ujino and Wadati [31], and a Rodrigues–type formula has been obtained [32], analogous to that recently given by Lapointe and Vinet [18] for Jack polynomials. Also Polychronakos [28] has recently considered the monomial expansion of $H_{(1^k)}(x;\alpha)$ and given its normalization. To our knowledge there have been no previous discussions of the generalized Laguerre polynomials in the context of the Calogero-Sutherland model.

Regarding the global limit of the density computed in Section 5, we know of no other works which consider this limit directly. However, using techniques from potential theory Johansson [15] has recently proved that, in the Hermite case, for all $\beta \geq 0$

$$\lim_{N\to\infty} \sqrt{\frac{2}{N}} \int_{-\infty}^{\infty} f(\sqrt{2N}x)\rho(\sqrt{2N}x)\,dx = \frac{2}{\pi}\int_{-1}^{1} f(x)\sqrt{1-x^2}\,dx.$$

(for $\beta = 2$ this result was first given in [3] using a mean-field approach) for any continuous, bounded f. The analogous result in the Jacobi case has also been obtained by Johansson [16]. These results establish that the smoothed density is that predicted by electrostatics, while our result establishes pointwise convergence to the electrostatic prediction.

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Appendix

In this appendix, the equations (4.6a), (4.6b) and (4.13) are derived as special cases of a p.d.e. satisfied by ${}_2\mathcal{F}_1^{(\alpha)}(a,b;c;x;y)$ (recall (4.3b)).

Proposition A.1

Let ${}_2\mathcal{F}_1^{(\alpha)}(a,b;c;x;y)$ be defined by (4.3b). This function satisfies the p.d.e.

$$D_1^{(x)}\,F + \left(c - \frac{N-1}{\alpha}\right) E_0^{(x)}\,F - \left(a+b-\frac{N-1}{\alpha}\right) E_2^{(y)}\,F - \eta_2^{(y)}\,F = ab p_1(y)\,F, \qquad \text{(A.1)}$$

where D_k, E_k are defined by (2.12) and $\eta_2 := \frac{1}{2}[D_2, E_2]$, and is in fact the unique solution of the equation of the form

$$F(x,y) = \sum_\kappa A_\kappa \frac{C_\kappa^{(\alpha)}(x) C_\kappa^{(\alpha)}(y)}{C_\kappa^{(\alpha)}(1^N)}, \qquad A_0 = 1 \qquad \text{(A.2)}$$

Proof We follow the method of Constantine and Muirhead[27] in the case $\alpha = 2$. With F given by (A.2), from (2.13) and (3.4) we have

$$\begin{aligned}
D_1^{(x)}\,F &= \sum_\kappa \sum_{i=1}^N \binom{\kappa^{(i)}}{\kappa} \left(\kappa_i + \frac{N-i}{\alpha}\right) \frac{C^{(\alpha)}(x)}{C^{(\alpha)}(1^N)}\, C_{\kappa^{(i)}}^{(\alpha)}(y)\, A_{\kappa^{(i)}} && \text{(A.3a)}\\
E_0^{(x)}\,F &= \sum_\kappa \sum_{i=1}^N \binom{\kappa^{(i)}}{\kappa} \frac{C^{(\alpha)}(x)}{C^{(\alpha)}(1^N)}\, C_{\kappa^{(i)}}^{(\alpha)}(y)\, A_{\kappa^{(i)}} && \text{(A.3b)}\\
E_2^{(y)}\,F &= \frac{1}{1+|\kappa|} \sum_\kappa \sum_{i=1}^N \binom{\kappa^{(i)}}{\kappa} \left(\kappa_i - \frac{i-1}{\alpha}\right) \frac{C^{(\alpha)}(x)}{C^{(\alpha)}(1^N)}\, C_{\kappa^{(i)}}^{(\alpha)}(y)\, A_\kappa && \text{(A.3c)}\\
\eta_2^{(y)}\,F &= \frac{1}{1+|\kappa|} \sum_\kappa \sum_{i=1}^N \binom{\kappa^{(i)}}{\kappa} \left(\kappa_i - \frac{i-1}{\alpha}\right) \left(\kappa_i - \frac{i-N}{\alpha}\right) \frac{C^{(\alpha)}(x)}{C^{(\alpha)}(1^N)}\, C_{\kappa^{(i)}}^{(\alpha)}(y)\, A_\kappa && \text{(A.3d)}\\
p_1(y)\,F &= \frac{1}{1+|\kappa|} \sum_\kappa \sum_{i=1}^N \binom{\kappa^{(i)}}{\kappa} \frac{C^{(\alpha)}(x)}{C^{(\alpha)}(1^N)}\, C_{\kappa^{(i)}}^{(\alpha)}(y)\, A_\kappa && \text{(A.3e)}
\end{aligned}$$

Substituting (A.3) in (A.1), equating coefficients of $C_\kappa^{(\alpha)}(x)/C_\kappa^{(\alpha)}(1^N)$ and then equating coefficients of $C_{\kappa^{(i)}}^{(\alpha)}(y)\binom{\kappa^{(i)}}{\kappa}$ gives

$$\left(c+\kappa_i-\frac{i-1}{\alpha}\right)A_{\kappa^{(i)}}=\frac{1}{1+|\kappa|}\left(a+\kappa_i-\frac{i-1}{\alpha}\right)\left(b+\kappa_i-\frac{i-1}{\alpha}\right)A_\kappa$$

This is a first order difference equation and so has a unique solution once the initial condition ($A_0=1$) is specified. It is straightforward to verify that the solution is

$$A_\kappa=\frac{1}{|\kappa|!}\frac{[a]_\kappa^{(\alpha)}[b]_\kappa^{(\alpha)}}{[c]_\kappa^{(\alpha)}}$$

The equation (4.13) for ${}_1\mathcal{F}_1^{(\alpha)}(a;c;x;y)$ follows from (A.1) by changing variables $y\to y/b$ and then taking $b\to\infty$. The equation (4.6b) for ${}_0\mathcal{F}_0^{(\alpha)}(x;y)$ follows from that for ${}_1\mathcal{F}_1^{(\alpha)}(a;c;x;y)$ by setting $a=c=(N-1)/\alpha$, while the equation (4.6a) follows from (4.13) (with a and c interchanged) for ${}_1\mathcal{F}_1^{(\alpha)}(a;c;x;y)$ by changing variables $y\to y/a$ and taking $a\to\infty$.

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