A unified representation-theoretic approach to special functions

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Dedicated to I.M.Gel'fand on the occasion of his 80th birthday

hep-th 9312101

A representation-theoretic approach to special functions was developed in the 40-s and 50-s in the works of I.M.Gelfand, M.A.Naimark, N.Ya.Vilenkin, and their collaborators (see [V],[VK]). The essence of this approach is the fact that most classical special functions can be obtained as suitable specializations of matrix elements or characters of representations of groups. Another rich source of special functions is the theory of Clebsch-Gordan coefficients which describes the geometric juxtaposition of irreducible components inside the tensor product of two representations (cf. [VK]). Finally, in recent works on representations of (quantum) affine Lie algebras it was shown that matrix elements of intertwining operators between certain representations of these algebras are interesting special functions – (q-)hypergeometric functions and their generalizations [TK, FR].

In this paper we suggest a general method of getting special functions from representation theory which unifies the three methods mentioned above and allows one to define and study many new special functions. We illustrate this method by a number of examples – Macdonald's polynomials, eigenfunctions of the Sutherland operator, Lamé functions. Other examples will be described in our future papers.

1. Vector-valued spherical functions. Let $\mathcal{H}$ be a Hopf algebra over $\mathbb{C}$, $H \subset \mathcal{H}$ be a subgroup of group-like elements, V, W, U be irreducible $\mathcal{H}$-modules. Let $\Phi : V \to W \otimes U$ be an intertwining operator for $\mathcal{H}$. From this data one can construct two kinds of special functions.

1. Vector-valued matrix elements. Set $f_{vw\Phi}(h) =< w, \Phi hv > \in U$, $h \in H$, $v \in V$,$w \in W^*$. We call this function on H a U-valued matrix element.

2. Vector-valued characters. Let $V = W$. Set $\chi_\Phi(h) = \mathrm{Tr}\,|_V\,(\Phi h) \in U$, $h \in H$. We call this function on H a U-valued character.

Example 1. If $U = \mathbb{C}$, $V = W$, $\Phi = \mathrm{Id}$, and we get the classical matrix elements and characters.

Example 2. The numbers $< f_{vw\Phi}(e), u >$, $u \in U^*$, where e is the identity element in H, are the Clebsch-Gordan coefficients for $\mathcal{H}$ (we assume that v, w, u are taken from some bases of V, W^*, U^*).

Example 3. Let $\mathcal{H}$ be the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ corresponding to a finite -dimensional simple Lie algebra $\mathfrak{g}$ (q can be equal to 1), $H = \{e\}$. Let V, W be Verma modules over $\mathcal{H}$ with the same central charge, and let $U = U_1(z_1) \otimes \cdots \otimes U_n(z_n)$, where $z_i \in \mathbb{C}^*$, U_i are finite-dimensional representations of $\mathcal{H}$, and $U_i(z)$ denotes the representation of $\mathcal{H}$ in the space U_i defined by $\pi_{U_i(z)}(a) = z^{\deg(a)}\pi_{U_i}(a)$, $a \in \mathcal{H}$, $\deg(a)$ being the homogeneous degree of a. Let $\Phi : V \to W \otimes U$ be a vertex (=intertwining) operator (here $\otimes$ should be understood as a completed tensor product). Let v, w be the vacuum vectors of V, W^*. (For more details see [FR]). Then the $U_1 \otimes \cdots \otimes U_n$-valued matrix element $f_{vw\Phi}(e)$ is a function of

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n complex variables $z_1, ..., z_n$. For $q = 1$ such functions arise in conformal field theory as correlation functions on the sphere; for $q \neq 1$, they appear as correlation functions of solvable lattice models in statistical mechanics. It is known that they satisfy the (classical or quantum) Knizhnik-Zamolodchikov (KZ) equations and express via hypergeometric (q-hypergeometric) functions and their generalizations. Thus, solutions of (quantum) KZ equations can be regarded as Clebsch-Gordan coefficients for (quantum) affine Lie algebras.

2. A generalized Peter-Weyl theorem. In this section $H = G$ is a finite group or a compact Lie group, and $\mathcal{H}$ is the convolution algebra of (generalized) functions on G. Notation: dg is the invariant probability measure on G; $R(G)$ is the set of irreducible unitary representations of G; $U \in R(G)$; $(\cdot,\cdot)_U$ is the inner product in U; $L^2(G,U)$ is the Hilbert space of U-valued L^2-functions on G with respect to dg, with the inner product $< f_1, f_2 >= \int_G (f_1(g), f_2(g))_U dg$; $L^2(G,U)^G$ is the subspace of $L^2(G,U)$ spanned by all functions χ such that $\chi(hgh^{-1}) = h\chi(g)$, $g, h \in G$.

For every $V \in R(G)$, fix an orthonormal basis B_V of V. For any $V, W \in R(G)$, let X_{VW} be the space of G-homomorphisms: $\Phi : V \to W \otimes U$. This is a Hilbert space since it is isomorphic to the space of G-invariants in $V^* \otimes W \otimes U$. Let $\mathcal{B}_{VW}$ be an orthonormal basis in X_{VW}. Let $f_{vw\Phi} \in L^2(G,U)$ be defined by $f_{vw\Phi}(g) = (w, \Phi g v)_W \in U$, $g \in G, v \in V, w \in W, \Phi \in X_{VW}$. We have $f_{vw\Phi}(gh) = f_{(hv)w\Phi}(g) = g f_{v(g^{-1}w)\Phi}(h)$. Also, define the functions $\chi_\Phi(g) = \mathrm{Tr}\,|_V\ (\Phi g) = \sum_{v \in B_V} f_{vv\Phi}(g)$, $\Phi \in X_{VV}$. Clearly, $\chi_\Phi \in L^2(G,U)^G$.

Theorem. *(i)* $\{(dim V\, dim W)^{1/2} f_{vw\Phi} : v \in B_V, w \in B_W, \Phi \in \mathcal{B}_{VW}, V, W \in R(G)\}$ *is an orthonormal basis of* $L^2(G,U)$.

(ii) $\{\chi_\Phi : \Phi \in \mathcal{B}_{VV}, V \in R(G)\}$ *is an orthonormal basis of* $L^2(G,U)^G$.

When $U = \mathbb{C}$, this theorem reduces to the standard Peter-Weyl theorem [K].

Remark. This theorem can be generalized to the case of quantum groups (cf. [EK1]).

3. Macdonald's polynomials as vector-valued characters. Let $G = SU(n)$, $\mathcal{H}$ be the algebra of functions on G, and let H be the Cartan subgroup of G (diagonal matrices). Macdonald's polynomials $P_\lambda(q,k,h)$ ($k \in \mathbb{Z}, h \in H$, λ is a dominant integral weight for G) are uniquely defined by the following properties:

1. $P_\lambda(q,k,\cdot)$ are trigonometric polynomials on H symmetric under the action of the Weyl group S_n.

2. For fixed q and k, $P_\lambda(q,k,\cdot)$ are orthogonal on H with respect to the weight $|\Delta|^2$, where $\Delta(q,k,h) = \prod_{m=0}^{k-1} \prod_{\alpha>0}(1 - q^m e^{<\alpha,\xi>})$, where $\xi \in \mathfrak{h} = \mathrm{Lie}H$ is such that $h = e^\xi$, and α runs over all positive roots of G.

3. $P_\lambda = \chi_\lambda + \sum_{\nu<\lambda} c_{\lambda\nu}\chi_\nu$, where ν is a dominant integral weight of G, and χ_ν is the character of the irreducible representation of G with highest weight ν, and $c_{\lambda\nu}$ are constants depending on q, k.

Example: $P_\lambda(q,1) = \chi_\lambda$.

In the special case $q = 1$, Macdonald's polynomials are called Jack's symmetric functions.

Let $V = L_\lambda$ be the finite-dimensional irreducible representation of G with highest weight λ. Let $U = S^{kn}\mathbb{C}^n$, where k is a positive integer, be a representation of G. Let Φ be a nonzero intertwining operator $V \to V \otimes U$. Such an operator exists iff $\lambda \geq$

$k\rho$, where ρ is the half-sum of positive roots of G, and if it exists, it is unique up to a factor. Let $W(h)$ be the Weyl denominator: $W(h) = \prod_{\alpha>0}(e^{\frac{1}{2}<\alpha,\xi>} - e^{-\frac{1}{2}<\alpha,\xi>})$, $h = e^{\xi}$. Let $\lambda = \nu + k\rho$, where ν is any dominant integral weight. Define the functions $\psi_\nu(k,h) = W(h)^{-k}\mathrm{Tr}|_V(\Phi h)$. These functions take values in the zero weight component of U, which is one-dimensional. Thus, we can regard them as scalar functions, choosing the normalization in such a way that the coefficient to $e^{<\nu,\xi>}$ in ψ_ν is 1.

Theorem. [EK1] *The functions $\psi_\nu(k,h)$ are the Jack's symmetric functions $P_\nu(1,k+1,h)$.*

Remark. The orthogonality of $\{\psi_\nu\}$ immediately follows from the generalized Peter-Weyl theorem.

In the special case $G = SU(2)$ the polynomials $P_\nu(1,k+1,h)$ are the classical Gegenbauer polynomials – the even trigonometric polynomials in one variable orthogonal with respect to the weight $\sin^{2k+2}x$.

Let now $\mathcal{H}$ be the quantum group $U_q(\mathfrak{sl}_n)$. The rest of notation is as above (the modules L_λ and $S^{kn}\mathbb{C}^n$ are the $\mathcal{H}$-modules obtained by q-deformation of the corresponding $SU(n)$-modules). Let $W_{q,m}(h) = \prod_{m=0}^{k-1}\prod_{\alpha>0}(q^m e^{\frac{1}{2}<\alpha,\xi>} - q^{-m}e^{-\frac{1}{2}<\alpha,\xi>})$, $h = e^{\xi}$. Define the functions $\psi_\nu(q,k,h) = W_{q,k}(h)^{-1}\mathrm{Tr}|_V(\Phi h)$, with the normalization described above.

Theorem.[EK1] *The functions $\psi_\nu(k,h)$ are the Macdonald's polynomials $P_\nu(q^2,k+1,h)$.*

4. Diagonalization of the Sutherland operator. The Sutherland operator is the Hamiltonian of a quantum many-body problem (see [OP]): $H = -\frac{1}{2}\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + C\sum_{i<j}\sinh(x_i - x_j)^{-2}$ (C is a constant). It is known [OP] that H has n functionally independent commuting quantum integrals $L_1, L_2, ..., L_n$: $L_m = \sum_{j=1}^n \frac{\partial^m}{\partial x_m}$ +lower order terms ($L_1 = \sum_{j=1}^n \frac{\partial}{\partial x_j}$, $L_2 = -2H$), and $[L_i, H] = 0$ for all i. Therefore, it makes sense to consider the system of differential equations $L_i\psi = \Lambda_i\psi$, $\Lambda_i \in \mathbb{C}$ (the eigenvalue problem). For each set of eigenvalues Λ_i this system has $n!$ linearly independent solutions. For special values of $\{\Lambda_i\}$ one of these solutions expresses via Jack's polynomials, but in general solutions are transcendental and express via generalized hypergeometric functions. Here we interpret these solutions as vector-valued characters for $\mathfrak{gl}_n$.

Let $\mathcal{H} = U(\mathfrak{gl}_n)$, $\mathfrak{h} = \mathbb{C}^n$ be the Cartan subalgebra in $\mathfrak{gl}_n$, $H = \exp(\mathfrak{h})$. Let V be the Verma module over $\mathfrak{gl}_n$ with highest weight $\lambda \in \mathbb{C}^n$. Let $\mu \in \mathbb{C}$ be such that $C = \mu(\mu+1)$, and let W be the module spanned by all functions of the form $z_1^{\mu+m_1}z_2^{\mu+m_2}\dots z_n^{\mu+m_n}$, $m_j \in \mathbb{Z}$, $\sum_j m_j = 0$, with the action of $\mathfrak{gl}_n$ given by $E_{ij} \mapsto z_i\frac{\partial}{\partial z_j} - \mu\delta_{ij}$ ($(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$). Let $\Phi : V \to V \otimes W$ be an intertwining operator for $\mathfrak{gl}_n$ ($\otimes$ is the completed tensor product). Φ is unique up to a factor if V is irreducible (which happens for a generic λ).

Define the functions $\phi_\lambda(x_1,...,x_n) = W(e^{\xi})^{-1}\mathrm{Tr}|_V(\Phi e^{\xi})$, where $\xi = 2\sum_{j=1}^n x_j E_{jj}$ (W is the Weyl denominator). As in Section 3, we regard ϕ_λ as scalar functions.

Theorem. [E] *(i) The function ψ_λ satisfies the system of equations $L_i\psi_\lambda = \Lambda_i\psi_\lambda$, where $\Lambda_i = p_i(\lambda+\rho)$, and p_m are symmetric polynomials with the highest term proportional to $\sum_j \lambda_j^m$.*

(ii) For a generic λ, the functions $\psi_{\sigma(\lambda+\rho)-\rho}$, $\sigma \in S_n$, form a basis in the space

of solutions of the system $L_i\psi = p_i(\lambda+\rho)\psi$.

Remark. This theorem can be generalized to the case of quantum $\mathfrak{gl}_n$, which provides a diagonal basis of functions for Macdonald's difference operators (cf. [Ch])

5. Lamé functions as vector-valued characters of $\widehat{sl}_2$. We preserve the notations of Example 3 from Section 1. Let $\mathfrak{g} = \mathfrak{sl}_2$, $V = W = M_{\lambda,k}$ is the Verma module with highest weight λ and central charge k, $n = 1$, $z_1 = 1$, and U_1 is the same as in Section 4. Consider the function

$$F(x,\tau) = \left(\mathrm{Tr}|_{M_{k/2,k}}(e^{-\pi i(2\tau d+(x+\frac{\tau}{2})h)})\right)^{-1} \mathrm{Tr}|_V(\Phi e^{-\pi i(2\tau d+(x+\frac{\tau}{2})h)}),$$

where d is the homogeneous gradation operator and $h = \mathrm{diag}(1,-1) \in \mathfrak{sl}_2$. This is a 1-point correlation function of conformal field theory on the torus. As in Sections 3,4, we regard F as a scalar function.

Theorem. [EK] *F satisfies the Schrödinger equation $-2\pi i(k+2)\frac{\partial F}{\partial \tau} + \frac{\partial^2 F}{\partial x^2} = \mu(\mu+1)(\wp(x+\frac{\tau}{2},\tau)+c)F$, where $\wp$ is the Weierstrass elliptic $\wp$-function, and c is a constant.*

If μ is a positive integer, it is possible to express F as a μ-dimensional integral of products of powers of theta-functions and exponents. When $k \to -2$ (critical level), the above equation becomes Lamé equation [WW]; finding the asymptotics of the integrals, we recover classical formulas from [WW], expressing Lamé functions via theta-functions (see [EK]).

Acknowledgements. We would like to thank our advisor Professor Igor Frenkel for stimulating discussions and Professors I.Cherednik, R.Howe, D. Kazhdan and A.Varchenko for helpful remarks.

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