Constructive Invariant Theory and the Linearization Problem

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Introduction

In this thesis, some problems in the theory of transformation groups will be discussed. The first chapter is about constructive invariant theory, and the other two chapters are dedicated to the algebraic and holomorphic linearization problem. The three chapters are independent from each other.

The first chapter is related to the fourteenth problem of Hilbert ([26]). If G is an algebraic group acting linearly on an n-dimensional vector space V (all defined over some field k), then G also acts on $\mathcal{O}(V) \cong k[X_1, \ldots, X_n]$, the ring of polynomial functions on V. If $\mathcal{O}(V)^G$ is the ring of invariant functions, then Hilbert's question can be formulated as:

Is the invariant ring $\mathcal{O}(V)^G$ always finitely generated as a k-algebra?

(the original formulation was more general). Nagata gave a counterexample in 1959 ([50]). Hilbert proved essentially that $\mathcal{O}(V)^G$ is finitely generated when G is linearly reductive in 1890 ([24]). This proof was criticized because the new revolutionary abstract methods of Hilbert were not constructive at all. Three years later he published another paper ([25]) with a different, more constructive proof. In this chapter Hilbert's first proof will be extended to make it constructive too, and an algorithm will be described to compute a set of generators of the invariant ring. Essentially this algorithm consists of a single application of Buchberger's algorithm to compute a Gröbner Basis. This algorithm is easy to implement and works for all linearly reductive groups. Using Hilbert's second proof, Popov found an upper bound for the minimal d such that the invariants of degree $\leq d$ generate the invariant ring for semi-simple G in characteristic 0 (see [54] and [55]). Hiss improved this bound and generalized it to arbitrary linearly reductive groups (cf. [27], [11]). Her bound for primary invariants doesn't depend anymore on the dimension of the representation. In special cases, for example tori, finite groups, SL_2, better bounds are known. We will show that there is a relation between some conjectures in commutative algebra, and the problem of finding good degree bounds. If for example the Eisenbud-Goto Conjecture is true (Conjecture I.4.1), then one can proof a good degree bound for generating invariants which would drastically improve the bounds of Popov and Hiss. A classical result of Noether is that for finite groups in characteristic 0, the ring of invariants is generated by invariants of degree $\leq |G|$, where $|G|$ is the order of the group (cf. [52]). It was conjectured that this is also the case in positive characteristic whenever the characteristic doesn't divide $|G|$. This can be shown if the Subspaces Conjecture is true (Conjecture I.4.2).

The second and the third chapter deal with the linearization problem. Suppose that G is a reductive linear algebraic group over $\mathbb{C}$, acting algebraically on $\mathbb{C}^n$. Is every G-action on $\mathbb{C}^n$ essentially a linear action? Or more precise: Does there exist for every G-action on $\mathbb{C}^n$ a polynomial automorphism, such that the action will become linear if we conjungate the action with this automorphism. In 1989 the first counterexamples were found by Schwarz (see [60]). He used algebraic G-vector bundles over representations to construct counterexamples for SL_2 and O_2. In fact, Knop showed there are counterexamples for all non-abelian connected reductive groups (cf. [38]). This approach doesn't work for abelian groups, because of a result of Masuda, Moser-Jauslin and Petrie ([47]), stating that G-vector bundles over representations are trivial whenever G is abelian and reductive. There are no examples known of non-linearizable algebraic actions of abelian reductive groups. For the multiplicative group $\mathbb{C}^\star$ there are a few positive results known (see [40]). All $\mathbb{C}^\star$-actions on $\mathbb{C}$ and $\mathbb{C}^2$ are linearizable (see for example [5] and [6]). Kaliman, Koras, Makar-Limanov and Russell are close to proving that all $\mathbb{C}^\star$ actions on $\mathbb{C}^3$ are linearizable. The $\mathbb{C}^\star$ linearization problem on $\mathbb{C}^3$ is related to examples of "exotic algebraic structures on $\mathbb{C}^3$". There are examples of hypersurfaces in $\mathbb{C}^4$ which are contractible, having a "hardcase" $\mathbb{C}^\star$ action. By a result of Dimca it follows from contractibility that the hypersurfaces are diffeomorphic to $\mathbb{R}^6$ (cf. [12]). One can show that if the hypersurface is algebraically isomorphic to $\mathbb{C}^3$, then the $\mathbb{C}^\star$-action cannot be linearizable. The first examples of contractible hypersurfaces were found by Libgober in a different context ([45]). His list was generalized by Dimca ([12]), Kaliman ([33]) and completed by Koras and Russell ([39]). In order to prove the $\mathbb{C}^\star$ linearization problem on $\mathbb{C}^4$ one has to proof that all the examples in Koras-Russell's list are not isomorphic to $\mathbb{C}^3$. The simplest example of the list of Koras and Russell is the hypersurface $x + x^2y + z^2 + t^3 = 0$. It has a $\mathbb{C}^\star$ action given by

$$\lambda \cdot (x, y, z, t) = (\lambda^6 x, \lambda^{-6} y, \lambda^3 z, \lambda^2 t).$$

If $x + x^2y + z^2 + t^3 = 0$ is isomorphic to $\mathbb{C}^3$, then this action cannot be linearizable. However, Makar-Limanov showed in [46] that $x + x^2y + z^2 + t^3 = 0$ is not isomorphic to $\mathbb{C}^3$. He defined an invariant of rings which distinguishes the rings $\mathbb{C}[x, y, z, t]/(x + x^2y + z^2 + t^3)$ and the polynomial ring $\mathbb{C}[x, y, z]$. Kaliman and Makar-Limanov showed that all hypersurfaces in the list of Koras and Russell are not isomorphic to $\mathbb{C}^3$ (cf. [34]). In the second chapter we give a short proof of the fact that $x + x^2y + z^2 + t^3 = 0$ is not isomorphic to $\mathbb{C}^3$. A new invariant is introduced to distinguish the coordinate rings.

Until very recently the linearization problem in the holomorphic category was open. Heinzner and Kutzschebauch showed that all holomorphic G-vector bundles over representations are trivial when G is a complex Lie group (see [22]). So the method of Schwarz cannot work in this situation. Kutzschebauch and I adapted an algebraic method of Asanuma (cf. [3], [4]) to the holomorphic category, to obtain non-linearizable holomorphic group actions. Asanuma constructed non-linearizable algebraic $\mathbb{R}^\star$-actions on $\mathbb{R}^5$ and non-linearizable algebraic $K^\star$-actions on K^4 where K is a field of positive characteristic. For example, a non-linearizable $\mathbb{R}^\star$-action on $\mathbb{R}^5$ can be obtained from *non-rectifiable* embeddings of $\mathbb{R}$ in $\mathbb{R}^3$, in the sense that there doesn't exist any polynomial automorphism of $\mathbb{R}^3$ which maps the image of $\mathbb{R}$ onto a straight line. Chapter III is basically a paper jointly written with Frank Kutzschebauch. In this chapter will be explained how one can obtain non-linearizable holomorphic $\mathbb{C}^\star$-actions using non-rectifiable proper holomorphic embeddings of $\mathbb{C}$ into $\mathbb{C}^n$, found by Forstneric, Globevnik, Rosay and Rudin (see [16], [56]). By extending the method of Asanuma, I was able to find counterexamples for all complex reductive groups.

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Chapter I

Computation of Reductive Group Invariants

1. Introduction

Let G be a linear algebraic group defined over the field k. Let $\rho : G \to \mathrm{GL}(V)$, be a representation of G on a k-vector space V of dimension $n < \infty$. The group G acts linearly on the k-algebra $\mathcal{O}(V) \cong k[X_1, X_2, \ldots, X_n]$ of polynomial functions on V, the *coordinate ring* of V. The *invariant ring* $\mathcal{O}(V)^G$ is the subalgebra of invariant functions. This invariant ring was a major object of research in the last century (see [48]). A natural question to ask is:

Is the invariant ring finitely generated as a k-algebra?

This question is essentially Hilbert's 14^{th} problem (see [26]), although his formulation was more general. In 1959 Nagata constructed a counterexample (see [50]). For reductive groups G, $\mathcal{O}(V)^G$ is finitely generated by results of Hilbert, Weyl, Mumford, Nagata, Haboush and others (see [49]).

The first positive result in this direction was proved by Gordan in 1868. He proved in [19] that $\mathcal{O}(V)^{\mathrm{SL}_2}$ is finitely generated for every finite dimensional SL_2-module V, using a constructive approach known as the *symbolic method*. Hilbert proved in 1890 that $\mathcal{O}(V)^G$ is always finitely generated when G is *linearly reductive* (see [24]). A group is called *linearly reductive* when every finite dimensional G-module is the direct sum of irreducible modules (see for example [65]). Examples of linearly reductive groups are:

- *reductive* groups if $\mathrm{char}(k) = 0$: semisimple groups, torus groups, finite groups, etc.;
- if $\mathrm{char}(k) > 0$: torus groups, finite groups G such that $\mathrm{char}(k)$ doesn't divide the group order $|G|$.

The proof of Hilbert's finiteness result was not constructive. When Gordan saw Hilbert's method he made his famous exclamation: "Das ist Theologie und nicht Mathematik".*

* This is theology and not mathematics.

After this criticism HILBERT wrote another paper in 1893 (see [25]) with a more constructive proof. He showed how to construct homogeneous algebraically independent $f_1, f_2, \ldots, f_r \in \mathcal{O}(V)^G$ such that $\mathcal{O}(V)^G$ is a finite $k[f_1, f_2, \ldots, f_r]$-module. These $f_1, f_2, \ldots, f_r$ are called *primary invariants* and the module generators of $\mathcal{O}(V)^G$ as a $k[f_1, f_2, \ldots, f_r]$-module are called *secondary invariants.* This method leads to algorithms as described in STURMFELS' book (see [67]). There are several implementations for computing invariant rings of finite groups. KEMPER wrote the INVAR package for the computer algebra system MAPLE (cf. [36], [37]). HEYDTMANN wrote the FINVAR.LIB library in SINGULAR (cf. [23]). There is also an implementation by GATERMANN in REDUCE (see [18]). In general, HILBERT's method can be complicated because one has to compute the normal closure of $k[f_1, f_2, \ldots, f_r]$ within the quotient field of $\mathcal{O}(V)$ which is a difficult computational problem.

We will show that the first proof of Hilbert can be extended to make it constructive too. This will lead to an algorithm for computing generators of the invariant ring for arbitrary linearly reductive groups G. This algorithm is essentially a single Gröbner Basis computation.

Suppose now that G is a linearly reductive group. An other interesting problem is to find upper bounds for the degrees of the generators of the invariant ring. We define the following constant:

$$\beta_G(V) = \min\{\, d \mid \mathcal{O}(V)^G \text{ is generated by invariants of degree} \leq d\,\}.$$

For some special cases, a good bound is known. JORDAN gave a good bound in the case $G = \mathrm{SL}_2$ and $\mathrm{char}(k) = 0$, using the methods of GORDAN (see [30] and [31]). Let $V_d := k[X, Y]_d$ be the vector space of homogeneous polynomials in two variables of degree d. The group SL_2 acts by substitution on V_d. These are all irreducible SL_2-modules. If V is an SL_2-module, then we can write $V = V_{d_1} \oplus V_{d_2} \oplus \cdots \oplus V_{d_t}$. JORDAN proved that

$$\beta_{\mathrm{SL}_2}(V) \leq d^6 \qquad \text{where } d = \max\{2, d_1, d_2, \ldots, d_t\}.$$

For finite groups we have the bound of NOETHER (see [52]): If G is a finite group and $\mathrm{char}(k) > |G|$ or $\mathrm{char}(k) = 0$, then

$$\beta_G(V) \leq |G|.$$

In special cases, SCHMID improved this bound (cf. [58], [59]). If $\mathrm{char}(k)$ does not divide $|G|$ then one still has the weaker bound

$$\beta_G(V) \leq \max\{|G|, n(|G| - 1)\}.$$

SMITH proved that one still has NOETHER's bound in this case if G is solvable (cf. [62]).

We will show that if the Subspace Conjecture (conjecture 4.2) is true, then Noether's bound holds whenever G is finite and linearly reductive, i.e., if $\mathrm{char}(k)$ *does not divide the group order.*

If $\mathrm{char}(k)$ is arbitrary, then NOETHER's bound does not hold in general. GÖBEL gave in [20] a constructive proof of the bound

$$\beta_G(V) \leq \max\{n, n(n-1)/2\}$$

if V is a permutation representation.

Suppose $G := T$ is a r-dimensional torus acting on V. Let $X^\star(T) \cong \mathbb{Z}^r$ be the group of weights. Choose a translation invariant volume form on $E := X^\star(T) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^r$such that the lattice $X^\star(T) \subset E$ has covolume 1. The action of T on V can be assumed to be diagonal, and it acts with n weights $\chi_1, \chi_2, \ldots, \chi_n \in X^\star(T)$. Let $\mathcal{C}_V \subset E$ be the convex hull of $\chi_1, \ldots, \chi_n$. WEHLAU proved in [68] the following bound:

$$\beta_T(V) \leq \max\{n - r - 1, 1\}\, r!\, \mathrm{vol}(\mathcal{C}_V).$$

The first general bound for semi-simple groups was found by POPOV (see [54] and [55]). He combined the second constructive proof of HILBERT with the results of HOCHSTER and ROBERTS stating that the invariant ring is Cohen-Macaulay (see [28]). In her thesis, HISS was able to improve this bound and generalize it to arbitrary connected reductive groups, using some ideas of KNOP's (see [27], [11]). Her degree bound for primary invariants does not depend on the dimension of the representation anymore. Suppose that G is a m-dimensional connected reductive group, let $U \subset G$ be a maximal unipotent subgroup and let $T \subseteq G$ be a maximal torus of rank r . Let $\mathcal{C}_V$ be as before and we define N_V by

$$N_V := \min\{\, \ell \mid X^{\ell+1} v = 0 \text{ for all } v \in V, X \in \mathrm{Lie}(U)\,\}.$$

Then we have

$$\beta_G(V) \leq n.\, \mathrm{lcm}\{1, 2, 3, \ldots, \sigma(V)\}$$

where

$$\sigma(V) \leq c(G) N_V^{m-r}\, \mathrm{vol}(\mathcal{C}_V) \qquad \text{and} \qquad c(G) := \frac{2^r (m+1)!\, r!}{(\frac{m-r}{2})!^2}.$$

This bound seems far from sharp. For example if $G = \mathrm{SL}_2$ and $V = V_d$ then we get $\sigma(V) \leq 96d^3$ and $\beta_G(V) \leq d.\, \mathrm{lcm}\{1, 2, \ldots, 96d^3\}$. This bound is of course much worse than the bound $\beta_G(V) \leq d^6$ of JORDAN.

Choose some norm $\|\cdot\|$ on $E = X^\star(T) \otimes_{\mathbb{Z}} \mathbb{R}$ and let

$$L_V = \max\{\, \|\chi_1\|, \|\chi_2\|, \ldots, \|\chi_n\|\,\}$$

where $\chi_1, \ldots, \chi_n$ are the weights of T appearing in V.

We will show that if the Eisenbud-Goto Conjecture is true (conjecture 4.1), then for any connected linearly reductive group there exist constants $C, C' > 0$ depending only on G and $\|\cdot\|$ such that

$$\beta_G(V) \leq \min\{C(nL_V)^m, C' L_V^{m^2+m}\}.$$

This would be a great improvement of the bounds of Popov and Hiss.

For a more detailed overview about constructive invariant theory, see [11].

2. The nullcone

Let k be a field, and G be a linearly reductive group over k. Suppose that V is an n-dimensional representation of G. The inclusion $\mathcal{O}(V)^G \hookrightarrow \mathcal{O}(V)$ corresponds to a morphism $\pi : V \to V/\!\!/G$, where $V/\!\!/G := \operatorname{Spec} \mathcal{O}(V)^G$ is the categorical quotient. The ring $\mathcal{O}(V) \cong k[X_1, X_2, \ldots, X_n]$ is graded $\mathcal{O}(V) = \bigoplus_{i\geq 0} \mathcal{O}(V)^i$, and this induces a grading on $\mathcal{O}(V)^G$. Let us write $\mathfrak{m}$ for the maximal homogeneous ideal $(X_1, X_2, \ldots, X_n)$ of $\mathcal{O}(V)$. We define $I_{\mathcal{N}}$ to be the ideal of $\mathcal{O}(V)$ generated by all homogeneous invariants of positive degree, which we will call the *zero-fiber ideal.* The space $\operatorname{Spec}(\mathcal{O}(V)/I_{\mathcal{N}})$ is equal to $\pi^{-1}\pi(0)$ in the schematic sense. The zero set of $I_{\mathcal{N}}$ is the so-called *nullcone* $\mathcal{N}$. The well-known Finiteness Theorem of Hilbert states that

Theorem 2.1. *If G is a linearly reductive group, and V is a representation of G, then $\mathcal{O}(V)^G$ is finitely generated.*

Proof. The proof of this theorem is based on the following two facts:

1. The polynomial ring over a field is noetherian: every ideal in the polynomial ring $\mathcal{O}(V)$ is finitely generated (Hilbert's Basissatz).
2. There exists a unique G-invariant linear projection $\mathcal{R} : \mathcal{O}(V) \to \mathcal{O}(V)^G$ (i.e., $\mathcal{R}(g \cdot f) = \mathcal{R}(f)$ for all $g \in G$, $f \in \mathcal{O}(V)$ and $\mathcal{R}(f) = f$ for all $f \in \mathcal{O}(V)^G$), which is called the *Reynolds operator* (see [65]). It has the following properties:
   (a) $\mathcal{R}$ is a $\mathcal{O}(V)^G$-module homomorphism;
   (b) if $W \subseteq \mathcal{O}(V)$ is a G-invariant linear subspace, then $\mathcal{R}(W) = W^G$.

The ideal $I_{\mathcal{N}}$ is generated by invariants, and by the noetherian property we only need finitely many of them to generate $I_{\mathcal{N}}$, say $I_{\mathcal{N}} = (f_1, f_2, \ldots, f_r)$. Suppose that h is a homogeneous invariant of degree d. Using induction on d, we proof that $h \in k[f_1, f_2, \ldots, f_r]$. If $d = 0$, then $h \in k$ and the statement is clear. Otherwise we can write

$$h = \sum_{i=1}^{r} a_i f_i$$

with $a_1, a_2, \ldots, a_r \in \mathcal{O}(V)$. Without loss of generality we may assume that a_i is homogeneous of degree $d - \deg f_i < d$ for all i. Applying the Reynolds operator yields

$$h = \mathcal{R}h = \mathcal{R}\left(\sum_{i=1}^{r} a_i f_i\right) = \sum_{i=1}^{r} \mathcal{R}(a_i) f_i.$$

Because the vector space of homogeneous polynomials of degree $\deg(a_i)$ is G-stable, we have that $\mathcal{R}(a_i)$ is homogeneous of degree $\deg(a_i) < d$ for all i. By induction hypothesis, $\mathcal{R}(a_1), \ldots, \mathcal{R}(a_r) \in k[f_1, f_2, \ldots, f_r]$, so $h \in k[f_1, \ldots, f_r]$. $\square$

The proof of Hilbert was criticized at the end of the nineteenth century because choosing the invariant generators $f_1, f_2, \ldots, f_r$ of $I_{\mathcal{N}}$ is a non-constructive step. However, we will show that there is a constructive way to obtain these $f_1, f_2, \ldots, f_r$.

Lemma 2.2. *Suppose $h_1, h_2, \ldots, h_s \in \mathcal{O}(V)$ are homogeneous and generate $I_{\mathcal{N}}$. Then $\mathcal{R}(h_1), \mathcal{R}(h_2), \ldots, \mathcal{R}(h_s)$ generate $\mathcal{O}(V)^G$ as a k-algebra.*

Proof. If $f_1, f_2, \ldots, f_r \in \mathcal{O}(V)^G$ are generators of $I_{\mathcal{N}}$, then the images of $f_1, \ldots, f_r$ span the vector space $I_{\mathcal{N}}/\mathfrak{m}I_{\mathcal{N}}$. Therefore, the action of G on $I_{\mathcal{N}}/\mathfrak{m}I_{\mathcal{N}}$ is trivial, so the Reynolds operator is equal to the identity on $I_{\mathcal{N}}/\mathfrak{m}I_{\mathcal{N}}$. The image of h_i and $\mathcal{R}(h_i)$ in $I_{\mathcal{N}}/\mathfrak{m}I_{\mathcal{N}}$ are the same for all i. So the images of the $\mathcal{R}(h_i)$ span $I_{\mathcal{N}}/\mathfrak{m}I_{\mathcal{N}}$ and $\mathcal{R}(h_1), \mathcal{R}(h_2), \ldots \mathcal{R}(h_s)$ generate $I_{\mathcal{N}}$ by the homogeneous Nakayama Lemma. Now we follow the proof of Hilbert's Finiteness Theorem to conclude $\mathcal{O}(V)^G = k[\mathcal{R}(h_1), \mathcal{R}(h_2), \ldots, \mathcal{R}(h_s)]$. □

3. Finding generators of the zero-fiber ideal

If generators of the $I_{\mathcal{N}}$ are known, then generators of the ring of invariants are found by applying the Reynolds operator by Lemma 2.2. In this section we deal with the problem of finding generators of $I_{\mathcal{N}}$. Consider the map

$$\psi : G \times V \to V \times V$$

given by $\psi(g, v) = (v, g \cdot v)$. Let $B \subseteq V \times V$ be the image of ψ. The ideal $\mathfrak{b}$ of the Zariski-closure $\overline{B}$ of B is given by:

$$\mathfrak{b} = \{\, h \in \mathcal{O}(V \times V) \mid h(v, g \cdot v) = 0 \, \forall g \in G, v \in V \,\}.$$

We will identify $\mathcal{O}(V \times V) \cong \mathcal{O}(V) \otimes \mathcal{O}(V)$ with the polynomial ring $k[X, Y]$, where X and Y are abbreviations of $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$.

Theorem 3.1. *We have an equality:*

$$((Y_1, Y_2, \ldots, Y_n) + \mathfrak{b}) \cap k[X] = I_{\mathcal{N}}.$$

Proof. "$\supseteq$": If $f(X) = f(X_1, \ldots, X_n)$ is an invariant of positive degree, then we get $f(X) = (f(X) - f(Y)) + f(Y)$. Now $f(Y) \in (Y_1, \ldots, Y_n)$ and $f(X) - f(Y) \in \mathfrak{b}$ because $(f(X) - f(Y))(v, g \cdot v) = f(v) - f(g \cdot v) = 0$ for all $g \in G$, $v \in V$.
"$\subseteq$": Conversely, if $f(X) \in (Y_1, \ldots, Y_n) + \mathfrak{b}$, then we can write

$$f(X) = \sum_i c_i(X) f_i(Y) + b(X, Y) \tag{3.1}$$

where $c_i(X) \in k[X]$, $f_i(Y) \in (Y_1, \ldots, Y_n)$ for all i and $b(X, Y) \in \mathfrak{b}$. We will view $V \times V$ as a G-variety, where G acts only on the second factor. The corresponding Reynolds operator:

$$\mathcal{R}_Y : k[X, Y] \to k[X, Y]^G = k[Y]^G[X]$$

is a $k[X]$-module homomorphism and its ristriction to $k[Y]$ is equal to the Reynolds operator $k[Y] \to k[Y]^G$ because of its uniqueness. We apply $\mathcal{R}_Y$ on (3.1):

$$f(X) = \sum_i c_i(X) \mathcal{R}_Y(f_i(Y)) + \mathcal{R}_Y(b(X, Y)). \tag{3.2}$$

Now $\mathcal{R}_Y(b(X, Y)) \in \mathfrak{b}$ because $\mathfrak{b}$ is G-stable (we use property (b) of the Reynolds operator). Let $\Delta : V \hookrightarrow V \times V$ be the diagonal morphism. The corresponding algebra homomorphism is $\Delta^\star : k[X, Y] \to k[X]$, $p(X, Y) \mapsto p(X, X)$. Applying $\Delta^\star$ to the equation (3.2) yields

$$f(X) = \sum_i c_i(X) \mathcal{R}(f_i(X)).$$

In fact, we have $\Delta^\star(\mathcal{R}_Y(b(X, Y))) = 0$, because $\mathcal{R}_Y(b(X, Y)) \in \mathfrak{b} \subseteq \ker(\Delta^\star)$. Now the $\mathcal{R}(f_i(X))$ are homogeneous invariants of positive degree, and we conclude that $f(X)$ lies in $I_{\mathcal{N}}$. □

We can reformulate Theorem 3.1 as:

Corollary 3.2. *If the ideal* $\mathfrak{b}$ *of* $\overline{B}$ *is generated by the* $f_1(X,Y), f_2(X,Y), \dots, f_r(X,Y)$, *then* $f_1(X,0), f_2(X,0), \dots, f_r(X,0)$ *generate* $I_{\mathcal{N}}$.

The geometric interpretation of Theorem 3.1 is

$$\overline{B} \cap (V \times \{0\}) = \mathcal{N} \times \{0\}.$$

It is important to notice that this equality holds in the schematic sense.

REMARK 3.3. Theorem 3.1 can be generalized. If $(h_1(X), \dots, h_r(X)) \subseteq k[X]$ is a G-invariant ideal, then

$$\big((h_1(Y), \dots, h_r(Y)) + \mathfrak{b}\big) \cap k[X]$$

is the ideal of $k[X]$ generated by $(h_1(X), \dots, h_r(X)) \cap k[X]^G$. Let $W \subseteq V$ is the zero set of $h_1, \dots, h_r$, $\pi : V \rightarrow: V/\!\!/G$ is the categorical quotient and $p_1 : V \times V \to V$ is the projection onto the first factor. Then the geometric interpretation of this generalization is

$$\overline{p_1((V \times W) \cap \overline{B})} = \pi^{-1}\pi(W).$$

A similar statement is true if V is an arbitrary affine G-variety.

4. Degree bounds for generating invariants

Let $\beta_G(V)$ be as in the introduction. Our aim is to give a good upper bound for $\beta_G(V)$. We first state two conjectures.

Conjecture 4.1 (EISENBUD-GOTO). *If* $\mathfrak{p}$ *is a homogeneous prime ideal in the polynomial ring* $k[X_1, \dots, X_n]$ *of degree* d, *then the module of* i*-th syzygies can be generated by homogeneous polynomials of degree*

$$\leq d + i + \dim_k \mathfrak{p}_1 - \operatorname{height} \mathfrak{p} + 1$$

where $\mathfrak{p}_1$ *is the vector space of linear functions in* $\mathfrak{p}$ *(see [14]).*

Conjecture 4.2 (Subspaces Conjecture). *Suppose that* W *is a* k*-vector space and* $W_1, W_2, \dots, W_d \subset W$ *are subspaces. If* I *is the ideal of* $W_1 \cup W_2 \cup \dots \cup W_d$, *then* I *is generated by homogeneous polynomials of degree* $\leq d$.

If G is a finite group then G is linearly reductive if and only if $\operatorname{char}(k)$ does not divide $|G|$. NOETHER's bound $\beta_G(V) \leq |G|$ is known only in the cases $\operatorname{char}(k) = 0$ and $\operatorname{char}(k) > |G|$.

Proposition 4.3. *Suppose that the Subspaces Conjecture is true, and* G *is a linearly reductive finite group, i.e.,* $\operatorname{char}(k)$ *doesn't divide* $|G|$. *Then we have* $\beta_G(V) \leq |G|$.

PROOF. The image B of $\psi : G \times V \to V \times V$ is the union of at most $|G|$ subspaces of dimension n. The ideal $\mathfrak{b}$ of $B = \overline{B}$ is generated by polynomials of degree $\leq |G|$. Using the lemma below we conclude $\beta_G(V) \leq d$. □

Lemma 4.4. *If* $\mathfrak{b}$ *is generated by homogeneous polynomials of degree* $\leq d$ *then we have* $\beta_G(V) \leq d$.

PROOF. If $\mathfrak{b}$ is generated by $f_1(X,Y),\ldots,f_r(X,Y)\in k[X,Y]$, then $I_{\mathcal{N}}$ is generated by $f_1(X,0),\ldots,f_r(X,0)$ by Corollary 3.2 and the ring $\mathcal{O}(V)^G$ is generated by the invariants $\mathcal{R}(f_1(X,0)),\ldots,\mathcal{R}(f_r(X,0))$ by Lemma 2.2. If $\mathfrak{b}$ is generated by homogeneous polynomials of degree $\leq d$, then so is the invariant ring. □

Suppose that G is a linearly reductive m-dimensional group, and V is an n-dimensional G-module. Choose a maximal torus $T\subseteq G$. Now $X^\star(T)=\hom(T,k^\star)\cong\mathbb{Z}^r$ be the group of characters. Choose some norm $||\cdot||$ on $E:=X(T)\otimes_{\mathbb{Z}}\mathbb{R}\cong\mathbb{R}^r$. The action of T on V can be assumed to be diagonal, and it acts with n weights, $\chi_1,\chi_2,\ldots,\chi_n\in X(T)$. Let

$$L_V=\max\{||\chi_1||,||\chi_2||,\ldots,||\chi_n||\}.$$

Theorem 4.5. *If the* EISENBUD-GOTO *Conjecture is true, G is a linearly reductive m-dimensional group and* $\mathrm{char}(k)=0$ *then there exist constants $C,C'>0$, depending only on the group G and the choice of $\|\cdot\|$, such that*

$$\beta_G(V)\leq\min\{\,C(nL_V)^m,C'L_V^{m^2+m}\,\}.$$

From now on we suppose $\mathrm{char}(k)=0$ and k algebraically closed. The variety $\overline{B}\subset V\times V$ is a cone. Therefore we can view $\overline{B}$ as a projective variety in $\mathbb{P}(V\times V)$ with some degree d. If G is connected, then $\overline{B}$ is irreducible and $\mathfrak{b}$ is a prime ideal. If the EISENBUD-GOTO Conjecture is true, then $\beta_G(V)\leq d$ by Lemma 4.4. So we need a bound for d. For this we will use the formula of KAZARNOVSKII.

We need some more notation. We fix a Borel subgroup B containing T. We denote by $\alpha_1,\alpha_2,\ldots,\alpha_{(m-r)/2}$, the positive roots. Let $\mathcal{W}$ be the Weyl group and let $e_1,e_2,\ldots,e_r$ be the Coxeter exponents, i.e., $e_1+1,e_2+1,\ldots,e_r+1$ are the degrees of the generating invariants of $\mathcal{W}$. We denote by $\mathcal{C}_V\subset E:=X^\star(T)\otimes_{\mathbb{Z}}\mathbb{R}$ the convex hull of the weights appearing in V, $\chi_1,\chi_2,\ldots,\chi_n$. On E we use the volume form dV given by any isomorphism $E\cong\mathbb{R}^r$ which identifies $X^\star(T)$ with $\mathbb{Z}^r$. Finally we fix a $\mathcal{W}$-invariant scalar product $(\cdot,\cdot)$ on E and denote, for any $\gamma\in E$, by $\hat\gamma\in E^\star$ the dual element defined by $\hat\gamma(\alpha)=2(\alpha,\gamma)/(\gamma,\gamma)$.

For a representation $\rho:G\to\mathrm{GL}(V)$, let $\delta_G(V)$ be the number of solutions $g\in G$ of

$$S_1(\rho(g))=S_2(\rho(g))=\ldots=S_m(\rho(g))=0 \tag{4.1}$$

for general $S_1,S_2,\ldots,S_m\in\mathrm{End}(V)^\star$.

Theorem 4.6 (KAZARNOVSKII, [35]). *The number $\delta_G(V)$ is equal to*

$$\frac{m!}{|\mathcal{W}|(e_1!e_2!\cdots e_r!)^2}\int_{\mathcal{C}_V}(\hat\theta_1\hat\theta_2\ldots\hat\theta_{\frac{m-r}{2}})^2. \tag{4.2}$$

where $\theta_1,\theta_2,\ldots,\theta_{\frac{m-r}{2}}$ are the positive roots.

See [11] for a more algebraic proof and [7] for a generalization of this result.

Corollary 4.7. *There exists a constant $C>0$ such that*

$$\delta_G(V)\leq CL_V^m.$$

PROOF. The homogeneous polynomial

$$f := (\hat{\theta}_1 \hat{\theta}_2 \ldots \hat{\theta}_{\frac{m-r}{2}})^2$$

has degree $m-r$. So $|f(z)| \leq C' \|z\|^{m-r}$ for some $C' > 0$. If we integrate f over a ball of radius L_V, then the result will be $\leq (2L_V)^r . C' L_V^{m-r}$, so

$$\delta_G(V) \leq \frac{m!\, 2^r}{|\mathcal{W}|(e_1! e_2! \cdots e_r!)^2} C' L_V^m = C L_V^m .$$

$\square$

Lemma 4.8. *If $S_1, S_2, \ldots, S_m$ are not in general position, $|\rho^{-1}(k^\star.\mathrm{id})| < \infty$ and the number of solutions of (4.1) is finite, then this number is still bounded from above by (4.2).*

PROOF. Let $\widehat{G} \subset \mathbb{P}(\mathrm{End}(V))$ be the projective variety given by the cone $k^\star.\rho(G)$. The number of solutions of (4.1) equals the number of solutions of $p \in \widehat{G}$ of

$$S_1(p) = S_2(p) = \ldots = S_m(p) = 0$$

multiplied by $|\rho^{-1}(k^\star.\mathrm{id})|$. By a generalization of the Bézout Theorem by Fulton and McPherson (see 12.3 of [17] and II.2.2 of [61]) this number of solutions is maximal when $S_1, S_2, \ldots, S_m$ are in general position. $\square$

We return to the computation of an upper bound for d, the degree of $\overline{B} \subset V \times V$ viewed as a projective variety in $\mathbb{P}(V \times V)$. Observe that $B = G\Delta(V)$ where $\Delta(V)$ is the diagonal of $V \times V$ and G acts only on the second factor of $V \times V$. Now $\Delta(V)$ is the zero set of n linear functions $f_i = X_i - Y_i$ with $1 \leq i \leq n$. Define for $i = 1, 2, \ldots, m$ functions on $G \times (V^\star \times V^\star)^{n+m-1}$ by

$$P_i(g, h_1, h_2, \ldots, h_{n+m-1}) = \det(g.f_1, g.f_2, \ldots, g.f_n, h_i, h_{i+1}, \ldots, h_{n+i-1})$$

Lemma 4.9. *Suppose that the generic stabilizer of the G-action on V is finite. For $h_i \in V^\star \times V^\star$ general enough, fixed, the number of common solutions of the P_i is finite and $\geq d$.*

PROOF. The dimension of B is equal to $n+m$ because the generic stabilizer is finite. The number of intersection points of B and $h_1 = h_2 = \ldots = h_{n+m-1} = 0$ in $\mathbb{P}(V \times V)$ is exactly d if the h_i are general linear functions on $V \times V$. So d is equal to the number of $g \in G$ for which

$$(4.3) \qquad f_1(g.p) = f_2(g.p) = \ldots = f_n(g.p) = h_1(p) = h_2(p) = \ldots = h_{n+m-1}(p) = 0$$

has a solution $p \in \mathbb{P}(V \times V)$, divided by the order of the generic stabilizer. For every such g and $1 \leq i \leq m$ we have $P_i(g, h_1, h_2, \ldots, h_{n+m-1}) = 0$. So the P_i have at least d common solutions for general $h_1, \ldots, h_{n+m-1}$.

We now prove finiteness. Let $Z \subset G \times (V^\star \times V^\star)^{n+m-1}$ be the zero set of $P_1, P_2, \ldots, P_m$. The projection of Z onto G is surjective. For fixed $g \in G$ we will compute its fiber dimension. Set $\overline{h}_i = h_i + W \in (V^\star \oplus V^\star)/W$, where W is the span of all $g.f_i$. Then

$$\det(g.f_1, g.f_2, \dots, g.f_n, h_i, h_{i+1}, \dots, h_{n+i-1}) = \det(\overline{h}_i, \overline{h}_{i+1}, \dots, \overline{h}_{n+i-1})$$

The solution space for $(\overline{h}_1, \dots, \overline{h}_{n+m-1})$ for these equations has dimension $(n+m)(n-1)$ by Lemma 4.10 below. For every $\overline{h}_i$ we have an n-dimensional space of representatives h_i, so the fiber dimension is $(n+m-1)n+(n+m)(n-1) = 2n^2+2nm-2n-m$. The dimension of Z is $m+(2n^2+2nm-2n-m) = 2n^2+2nm-2n$. The generic fiber of the projection to $(V^\star \times V^\star)^{n+m-1}$ must have dimension $(n+m-1)2n-(2n^2+2nm-2n) = 0$, so the general fiber is finite. $\square$

Lemma 4.10. *The variety*

$$Y := \{(v_1, v_2, \dots, v_t) \in k^{u\times t} \mid \det(v_i, v_{i+1}, \dots, v_{i+u-1}) = 0\,(1 \le i \le t-u+1)\}$$

has dimension $(t+1)(u-1)$.

Proof. The variety Y is given by $t-u+1$ equations, so the codimension of Y is $\le t-u+1$. Let W be the $(t-u+1)$-dimensional subspace of $k^{u\times t}$ of all matrices

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_{t-u+1} & 0 & \dots & 0 & 0 \\ 0 & \lambda_1 & \dots & \lambda_{t-u} & \lambda_{t-u+1} & \dots & 0 & 0 \\ \vdots & & \ddots & & & \ddots & & \vdots \\ 0 & 0 & \dots & \lambda_1 & \lambda_2 & \dots & \lambda_{t-u+1} & 0 \\ 0 & 0 & \dots & 0 & \lambda_1 & \dots & \lambda_{t-u} & \lambda_{t-u+1} \end{pmatrix}$$

Now $\det(v_1, v_2, \dots, v_u) = 0$ gives us $\lambda_1^n = 0$, so $\lambda_1 = 0$. From the second equation $\det(v_2, \dots, v_{u+1}) = 0$ follows $\lambda_2 = 0$, etc., so $W \cap Y = \{0\}$. So Y is a complete intersection and its dimension is $tu - (t-u+1) = (t+1)(u-1)$. $\square$

Proof of Theorem 4.5. Let

$$\pi : G \to \mathrm{GL}\big(\textstyle\bigwedge^n(V \times V)\big) \subset \mathrm{End}\big(\textstyle\bigwedge^n(V \times V)\big)$$

be the representation of G on $\bigwedge^n(V \times V)$. Now $P_i(g)$ is equal to $Q_i(\pi(g))$ where $Q_i \in \mathrm{End}\big(\bigwedge^n(V \times V)\big)^\star$ is defined by

$$A \in \mathrm{End}\big(\textstyle\bigwedge^n(V \times V)\big) \to \big(A(f_1 \wedge \dots \wedge f_n)\big) \wedge h_i \wedge \dots \wedge h_{n+i-1} \in \textstyle\bigwedge^{2n}(V \times V) \cong k$$

We apply Corollary 4.7 for $\bigwedge^n(V \times V)$. The weights appearing in $\bigwedge^n(V \times V)$ are all $\sum_{j=1}^u \lambda_j$ with $u \le n$ and $\lambda_1, \dots, \lambda_u$ weights of linear independent T-eigenvectors $v_1, v_2, \dots, v_u$ in V. It follows that $L_{\wedge^n(V\times V)} \le nL_V$. The number of solutions of

$$Q_1(\pi(g)) = Q_2(\pi(g)) = \dots = Q_{m+n-1}(\pi(g))$$

is finite by Lemma 4.9. So this number is at most $\delta_G\big(\bigwedge^n(V \times V)\big)$ by Lemma 4.8. On the other hand, from Lemma 4.9 follows that the number of solutions is at least d. So we get $\beta_G(V) \le d \le C(nL_V)^m$.

If the generic stabilizer is not finite, take some fixed representation V' with trivial generic stabilizer. Then

$$\beta_G(V) \leq \beta_G(V \oplus V') \leq C_2((n+n')(\max\{L_V, L_{V'}\})^m \leq C_3(nL_V)^m$$

where $n' = \dim V'$ and $C_3 > 0$ is a large enough constant.

If G is not connected, then let G° be the connected component of $\mathrm{id} \in G$, which is a normal subgroup. Now $\mathcal{O}(V)^{G^\circ}$ is generated by homogeneous invariants of degree $\leq C_3(nL_V)^m$. From NOETHER's bound follows that $\mathcal{O}(V)^G = (\mathcal{O}(V)^{G^\circ})^{G/G^\circ}$ is generated by homogeneous invariants of degree $\leq |G/G^\circ| C_3(nL_V)^m \leq C_4(nL_V)^m$.

Suppose we have a decomposition

$$V = V_1^{a_1} \oplus V_2^{a_2} \oplus \ldots \oplus V_t^{a_t}$$

where the V_i are irreducible and different. By Weyl's theorem (see [69]) the generators of invariants of $\mathcal{O}(V)$ can found by polarizing invariants of $\mathcal{O}(V')$ where

$$V' = V_1^{\dim(V_1)} \oplus V_2^{\dim(V_2)} \oplus \ldots \oplus V_t^{\dim(V_t)}$$

In this situation we have $\beta_G(V) \leq \beta_G(V') \leq C_4(n'L_V)^m$ where $n' = \dim V'$. The number of different irreducible representations of highest weight λ where λ has length $\leq L_V$ is $\leq C_5 L_V^r$. These representations all have dimension $\leq C_6 L_V^{(m-r)/2}$ because of the *Weyl formula* (see 3.2.6 of [53]):

$$\dim V_\lambda = \prod_{i=1}^{\frac{1}{2}(m-r)} \frac{\hat{\theta}_i(\lambda+\rho)}{\hat{\theta}_i(\rho)} \qquad \text{where } \rho = \frac{1}{2}\sum_{i=1}^{\frac{1}{2}(m-r)} \theta_i \in E$$

This implies $n' \leq (C_5 L_V^r)\Big(C_6 L_V^{(m-r)/2}\Big)^2 \leq C_7 L_V^m$, and so we finally deduce

$$\beta_G(V) \leq C_4(n'L_V)^m \leq C_4(C_7 L_V^{m+1})^m = C_8 L_V^{m(m+1)}.$$

□

EXAMPLE 4.11. *Binary forms.*

Let $G = \mathrm{SL}_2(k)$. The set of weights can be identified with $\mathbb{Z}$. Let V_d is the vector space of binary forms of degree d. The highest weight of this SL_2-module is d, so we have $L = d$, $m = \dim G = 3$ and $n = \dim V_d = d+1$. Suppose that the EISENBUD-GOTO Conjecture is true. Because of Theorem 4.5 we get

$$\beta_G(V_d) \leq C((d+1)d)^3 < C_2 d^6$$

for some constant $C_2 > 0$. If $V = V_{d_1} \oplus V_{d_2} \oplus \cdots \oplus V_{d_t}$ with $d_i \leq d$ for all i, then by Theorem 4.5 we have

$$\beta_G(V) \leq C' d^{3^2+3} = C' d^{12}.$$

Indeed, JORDAN proved that in this case we even have

$$\beta_G(V) \leq d^6$$

for $d \geq 2$ (see [30] and [31]).

5. The algorithm

Suppose that $\mathcal{O}(G)$ is isomorphic to $k[Z_1, Z_2, \ldots, Z_s]/I_G$, where I_G is an ideal of $k[Z_1, Z_2, \ldots, Z_s]$ generated by $h_1, h_2, \ldots, h_r$. The linear action on k^n is given by

$$\begin{pmatrix} X_1^g \\ X_2^g \\ \vdots \\ X_n^g \end{pmatrix} = \begin{pmatrix} a_{1,1}(g) & a_{1,2}(g) & \ldots & a_{1,n}(g) \\ a_{2,1}(g) & a_{2,2}(g) & \ldots & a_{2,n}(g) \\ \vdots & \vdots & & \vdots \\ a_{n,1}(g) & a_{n,2}(g) & \ldots & a_{n,n}(g) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

for all $g \in G$, where $a_{i,j} \in k[Z_1, Z_2, \ldots, Z_s]$ ($a_{i,j}$ is determined up to elements of I_G). Let $\Gamma \subseteq G \times V \times V$ be the Zariski-closed subset $\{ (g, x, gx) \mid g \in G,\ x \in V \}$. The coordinate ring of Γ is

$$k[Z_1, Z_2, \ldots, Z_s, X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n]/I_\Gamma$$

where the ideal I_Γ is generated by I_G and all $Y_i - \sum_{j=1}^n a_{i,j} X_j$ with $i = 1, 2, \ldots, n$. Now $\overline{B}$ is the closure of the projection of $\Gamma \subseteq G \times V \times V$ onto $V \times V$. The ideal $\mathfrak{b}$ of $\overline{B}$ is equal to $I_\Gamma \cap k[X, Y]$. This intersection can be computed using Gröbner basis (see [1], [10], or [13] for an introduction to Gröbner Basis). Choose an admissible ordering on the monomials of $k[Z, X, Y]$ such that Z_i is bigger than any monomial in X and Y. Compute a Gröbner basis S of I_Γ with respect to this ordering. The set $S \cap k[X, Y]$ is a Gröbner bases of $\mathfrak{b} = I_\Gamma \cap k[X, Y]$ (with respect to the induced ordering on $k[X, Y]$). If this Gröbner basis is $\{f_1, f_2, \ldots, f_l\} \subset k[X, Y]$, then $\{f_1(X, 0), f_2(X, 0), \ldots, f_l(X, 0)\}$ generate $I_\mathcal{N}$ (by Corollary 3.2). And finally $\mathcal{R}(f_1(X, 0)), \mathcal{R}(f_2(X, 0)), \ldots, \mathcal{R}(f_l(X, 0))$ generate $k[X]^G$ (by Lemma 2.2).

Application of $\mathcal{R}$ is not a trivial computation. It depends on the group G. For example if G is finite then $\mathcal{R} : k[X] \to k[X]^G$ is given by

$$\mathcal{R}(f) = \frac{1}{|G|} \sum_{g \in G} f^g.$$

For a semi-simple group the map $\mathcal{R}$ can be computed using the action of the Lie-algebra on $k[X]$. We will not discuss this here.

The algorithm is as follows:

`input:` $s, n \in \mathbb{N},\ h_1, h_2, \ldots, h_r, a_{i,j} \in k[Z_1, Z_2, \ldots, Z_s]\ (1 \leq i, j \leq n)$
$I_\Gamma := \{Y_1 - \sum_{j=1}^n a_{1,j} X_j, \ldots, Y_n - \sum_{j=1}^n a_{n,j} X_j, h_1, \ldots, h_r\}$
$S :=$ `Gröbner`$(I_\Gamma, Z_1 > Z_2 > \ldots > Z_s > X_1 > \ldots > X_n > Y_1 > \ldots > Y_n)$
$\mathfrak{b} := S \cap k[X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n]$
$I_\mathcal{N} := \{$ `substitute`$(\{Y_1 = 0, Y_2 = 0, \ldots, Y_n = 0\}, f) \mid f \in \mathfrak{b} \}$
`output:` $\{\mathcal{R}(g) \mid g \in I_\mathcal{N}\}$

6. Examples

In this section we will look at some concrete examples.

Example 6.1. *$\mathcal{G}_m$-action with weights -5,-3,2,4.*

We take the multiplicative group $\mathcal{G}_m$ with coordinate ring $k[Z_1, Z_2]/(Z_1 Z_2 - 1)$. The group acts on the four-dimensional vector-space with weights $-5, -3, 2, 4$. The matrix of this representation is

$$\begin{pmatrix} Z_2^5 & 0 & 0 & 0 \\ 0 & Z_2^3 & 0 & 0 \\ 0 & 0 & Z_1^2 & 0 \\ 0 & 0 & 0 & Z_1^4 \end{pmatrix}$$

We use the algorithm of the previous section to compute $k[X_1, X_2, X_3, X_4]^{\mathcal{G}_m}$. Let $I_\Gamma \subseteq k[Z, X, Y]$ be the ideal of the graph Γ. It is given by

$$I_\Gamma = (Z_1 Z_2 - 1, Y_1 - Z_2^5 X_1, Y_2 - Z_2^3 X_2, Y_3 - Z_1^2 X_3, Y_4 - Z_1^4 X_4)$$

Using Gröbner basis we can compute $\mathfrak{b} = I_\Gamma \cap k[X, Y]$ which is equal to

$$\mathfrak{b} = (X_4 Y_3^2 - X_3^2 Y_4, X_1 X_3 Y_2 - X_2 Y_1 Y_3, X_1 X_4 Y_2 Y_3 - X_2 X_3 Y_1 Y_4,$$

$$X_1 X_2 X_4^2 - Y_1 Y_2 Y_4^2, X_2^2 X_3 X_4 - Y_2^2 Y_3 Y_4, X_2^2 X_4^2 Y_3 - X_3 Y_2^2 Y_4^2, X_1^2 X_4 Y_2^2 - X_2^2 Y_1^2 Y_4,$$

$$X_2^3 X_4 Y_1 - X_1 Y_2^3 Y_4, X_1^2 X_3 X_4^2 - Y_1^2 Y_3 Y_4^2, X_1 X_2 X_3^2 X_4 - Y_1 Y_2 Y_3^2 Y_4, X_2^2 X_3^3 - Y_2^2 Y_3^3,$$

$$X_1^2 X_4^3 Y_3 - X_3 Y_1^2 Y_4^3, X_2^3 X_3^2 Y_1 - X_1 Y_2^3 Y_3^2, X_1^2 X_3^3 X_4 - Y_1^2 Y_3^3 Y_4, X_1 X_2 X_3^4 - Y_1 Y_2 Y_3^4,$$

$$X_1^3 X_4^3 Y_2 - X_2 Y_1^3 Y_4^3, X_2^4 X_3 Y_1^2 - X_1^2 Y_2^4 Y_3, X_2^4 X_4^3 - Y_2^4 Y_4^3, X_1^2 X_3^5 - Y_1^2 Y_3^5,$$

$$X_2^5 Y_1^3 - X_1^3 Y_2^5, X_1^4 X_4^5 - Y_1^4 Y_4^5)$$

We find generators of $I_\mathcal{N}$ by substituting $Y = 0$. The null-cone ideal is generated by the elements

$$X_1^4 X_4^5, X_1^2 X_3^5, X_2^4 X_4^3, X_1 X_2 X_3^4, X_1^2 X_3^3 X_4,$$

$$X_2^2 X_3^3, X_1 X_2 X_3^2 X_4, X_1^2 X_3 X_4^2, X_2^2 X_3 X_4, X_1 X_2 X_4^2$$

We don't have to apply the Reynolds operator here, because these generators are already $\mathcal{G}_m$-invariant. In the torus case the algorithm presented here does the same as Algorithm 1.4.5. in [67].

Example 6.2. *The finite group S_3 acting on a direct sum of the irreducible 2-dimensional and the signum representation.*

Let ζ be the third root of unity. As an algebraic set (but not as a group), S_3 is isomorphic to the set $\{1, \zeta, \zeta^2\} \times \{-1, 1\}$ because they have the same cardinality. A bijection is given by $(1\,2)^j(1\,2\,3)^i \mapsto (\zeta^i, (-1)^j)$. So the coordinate ring of S_3 can be identified with $k[Z_1, Z_2]/(Z_1^3 - 1, Z_2^2 - 1)$. The representation is given by the matrix

$$\begin{pmatrix} (1+Z_2)/2 & (1-Z_2)/2 & 0 \\ (1-Z_2)/2 & (1+Z_2)/2 & 0 \\ 0 & 0 & Z_2 \end{pmatrix} \begin{pmatrix} Z_1 & 0 & 0 \\ 0 & Z_1^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} Z_1(1+Z_2)/2 & Z_1^2(1-Z_2)/2 & 0 \\ Z_1(1-Z_2)/2 & Z_1^2(1+Z_2)/2 & 0 \\ 0 & 0 & Z_2 \end{pmatrix}$$

The ideal I_Γ defining Γ is given by

$$I_\Gamma = (Z_1^3-1, Z_2^2-1, Y_1 - X_1Z_1(1+Z_2)/2 - X_2Z_1^2(1-Z_2)/2,$$

$$Y_2 - X_1Z_1(1-Z_2)/2 - X_2Z_1^2(1+Z_2)/2, Y_3 - X_3Z_2)$$

Eliminating Z_1 and Z_2 with Buchbergers Algorithm yields

$$\mathfrak{b} = (X_3^2 - Y_3^2, X_1X_2 - Y_1Y_2, X_1^3 + X_2^3 - Y_1^3 - Y_2^3,$$

$$X_3Y_1^3 - X_3Y_2^3 + 2X_2^3Y_3 - Y_1^3Y_3 - Y_2^3Y_3, X_2X_3Y_1^2 - X_1^2X_3Y_2 + X_2Y_1^2Y_3 - X_1^2Y_2Y_3,$$

$$X_1X_3Y_1^2 - X_2^2X_3Y_2 - X_1Y_1^2Y_3 + X_2^2Y_2Y_3, X_2^2X_3Y_1 - X_1X_3Y_2^2 + X_2^2Y_1Y_3 - X_1Y_2^2Y_3,$$

$$X_1^2X_3Y_1 - X_2X_3Y_2^2 - X_1^2Y_1Y_3 + X_2Y_2^2Y_3, X_2^3X_3 - X_3Y_2^3 + X_2^3Y_3 - Y_2^3Y_3,$$

$$X_2^4 - X_2Y_1^3 + X_1^2Y_1Y_2 - X_2Y_2^3)$$

We obtain the zero-cone ideal by substituting $Y = 0$:

$$I_{\mathcal{N}} = (X_3^2, X_1X_2, X_1^3 + X_2^3, X_2^3X_3, X_2^4)$$

Note that we don't need X_2^4 because $X_2^4 = X_2(X_1^3 + X_2^3) - X_1^2(X_1X_2)$. The first three generators are already invariant and $\mathcal{R}(X_2^3X_3) = \frac{1}{2}(X_2^3X_3 - X_1^3X_3)$. So we finally get

$$k[X_1, X_2, X_3]^{S_3} = k[X_3^2, X_1X_2, X_1^3 + X_2^3, X_2^3X_3 - X_1^3X_3]$$

Remark 6.3. See [36] and [18] for other methods to compute invariants of finite group actions. Kemper implemented his method in the Invar package in Maple. In [37] he deals with the case where the ground field has positive characteristic.

Example 6.4. *The group* SL_2 *acting on* $W \oplus W \oplus S^2W$ *where* W *is the standard 2-dimensional representation.*

We take the group SL_2 whose coordinate ring is $k[Z_1, Z_2, Z_3, Z_4]/(Z_1Z_4 - Z_2Z_3 - 1)$. The standard representation on the 2-dimensional space W is given by the matrix

$$\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}$$

The representation on S^2W is given by the matrix

$$\begin{pmatrix} Z_1^2 & 2Z_2Z_1 & Z_2^2 \\ Z_3Z_1 & Z_4Z_1 + Z_3Z_2 & Z_4Z_2 \\ Z_3^2 & 2Z_4Z_3 & Z_4^2 \end{pmatrix}$$

Now we will take the 7-dimensional vector space $V := W \oplus W \oplus S^2W$ and compute the invariants $k[X_1, X_2, \ldots, X_7]^{\mathrm{SL}_2}$. Let $I_\Gamma \subseteq k[Z, X, Y]$ be the ideal which describes the graph $\Gamma \subseteq G \times V \times V$. We have

$$I_\Gamma = (Z_1X_1 + Z_2X_2 - Y_1, Z_3X_1 + Z_4X_2 - Y_2, Z_1X_3 + Z_2X_4 - Y_3, Z_3X_3 + Z_4X_4 - Y_4,$$

$$Z_1^2X_5 + 2Z_1Z_2X_6 + Z_2^2X_7 - Y_5, Z_1Z_3X_5 + Z_2Z_3X_6 + Z_1Z_4X_6 + Z_2Z_4X_7 - Y_6,$$
$$Z_3^2X_5 + 2Z_3Z_4X_6 + Z_4^2X_7 - Y_7, -Z_2Z_3 + Z_1Z_4 - 1)$$

We choose an ordering on the monomials in $Z_1, \ldots, Z_4, X_1, \ldots, X_7, Y_1, \ldots, Y_7$ such that $Z_i >$ every monomial in X and Y and we compute the reduced Gröbner basis. In a computation in Singular this basis had 66 elements of which 18 lie in the ring $k[X, Y]$. These 18 polynomials generate the ideal $\mathfrak{b}$. The ideal $\mathfrak{b}$ has 9 minimal generators, namely

$$\mathfrak{b} = X_1^2X_7 - 2X_1X_2X_6 + X_2^2X_5 - Y_1^2Y_7 + 2Y_1Y_2Y_6 - Y_2^2Y_5,$$
$$X_1X_3X_7 - 2X_2X_3X_6 + X_2X_4X_5 + X_6Y_2Y_3 - X_6Y_1Y_4 - Y_1Y_3Y_7 + Y_2Y_3Y_6 + Y_1Y_4Y_6 - Y_2Y_4Y_5,$$
$$X_3^2X_7 - 2X_3X_4X_6 + X_4^2X_5 - Y_3^2Y_7 + 2Y_3Y_4Y_6 - Y_4^2Y_5,$$
$$X_3X_7Y_1 - X_4X_6Y_1 - X_1X_7Y_3 + X_2X_6Y_3 + X_4Y_1Y_6 - X_2Y_3Y_6 - X_4Y_2Y_5 + X_2Y_4Y_5,$$
$$X_3X_6Y_1 - X_4X_5Y_1 - X_1X_6Y_3 + X_2X_5Y_3 + X_3Y_1Y_6 - X_1Y_3Y_6 - X_3Y_2Y_5 + X_1Y_4Y_5,$$
$$X_3X_7Y_2 - X_4X_6Y_2 - X_1X_7Y_4 + X_2X_6Y_4 + X_4Y_1Y_7 - X_2Y_3Y_7 - X_4Y_2Y_6 + X_2Y_4Y_6,$$
$$X_3X_6Y_2 - X_4X_5Y_2 - X_1X_6Y_4 + X_2X_5Y_4 + X_3Y_1Y_7 - X_1Y_3Y_7 - X_3Y_2Y_6 + X_1Y_4Y_6,$$
$$X_2X_3 - X_1X_4 - Y_2Y_3 + Y_1Y_4, X_6^2 - X_5X_7 - Y_6^2 + Y_5Y_7$$

Substituting $Y = 0$ yields

$$I_{\mathcal{N}} = (X_1^2X_7 - 2X_1X_2X_6 + X_2^2X_5, X_1X_3X_7 - 2X_2X_3X_6 + X_2X_4X_5,$$
$$X_3^2X_7 - 2X_3X_4X_6 + X_4^2X_5, X_2X_3 - X_1X_4, X_6^2 - X_5X_7)$$

These are all invariants except $X_1X_3X_7 - 2X_2X_3X_6 + X_2X_4X_5$. If we apply the Reynolds-operator then we obtain

$$\mathcal{R}(X_1X_3X_7 - 2X_2X_3X_6 + X_2X_4X_5) = X_1X_3X_7 - X_2X_3X_6 - X_1X_4X_6 + X_2X_4X_5$$

The ring of invariants is therefore generated by

$$X_1^2X_7 - 2X_1X_2X_6 + X_2^2X_5, X_1X_3X_7 - X_2X_3X_6 - X_1X_4X_6 + X_2X_4X_5,$$
$$X_3^2X_7 - 2X_3X_4X_6 + X_4^2X_5, X_2X_3 - X_1X_4, X_6^2 - X_5X_7$$

Remark 6.5. The algorithm presented here is short and easy to implement. The computation of the Gröbner basis of I_Γ takes a lot of time, so it is recommended to use a fast computer algebra system. The algorithm is implemented in the computer algebra system Singular. The computation of each example took less than a second on a Sparcstation 20.

Chapter II

On the hypersurface $x + x^2y + z^2 + t^3 = 0$ in $\mathbb{C}^4$

1. Introduction

The hypersurface Y given by $x + x^2y + z^2 + t^3 = 0$ has been studied because it is closely related to the linearization conjecture for $\mathbb{C}^\star$ (see [40]):

Conjecture 1.1 (Linearization Conjecture for $\mathbb{C}^\star$-actions). *Every algebraic $\mathbb{C}^\star$-action on $\mathbb{C}^n$ is a linear action up to conjungation with an automorphism of $\mathbb{C}^n$.*

This conjecture is true for $n = 2$ (see for example [5] and [6]) and open for $n \geq 3$. However, Kaliman, Koras, Makar-Limanov and Russell are close to proving that $\mathbb{C}^\star$-actions on $\mathbb{C}^3$ are linearizable. In the attempt to prove this conjecture, they encountered examples of "exotic algebraic structures on $\mathbb{C}^3$". These examples are smooth contractible hypersurfaces in $\mathbb{C}^4$ with a "hardcase" $\mathbb{C}^\star$-action. The hypersurfaces are diffeomorphic to $\mathbb{R}^6$ by a result of Dimca (see [12]). If such a hypersurface is algebraically isomorphic to $\mathbb{C}^3$, then the $\mathbb{C}^\star$-action cannot be linearizable. The first examples were found by Libgober in a different context (cf. [45]). His list was generalized by Dimca ([12]), Kaliman ([33]) and completed by Koras and Russell ([39]). In order to prove the $\mathbb{C}^\star$-linearization conjecture on $\mathbb{C}^3$, one has to prove that these examples are not isomorphic to $\mathbb{C}^3$. The simplest example of the list of Koras and Russell is the hypersurface $Y : x + x^2y + z^2 + t^3 = 0$ in $\mathbb{C}^4$. The $\mathbb{C}^\star$-action is given by

$$\lambda \cdot (x, y, z, t) = (\lambda^6 x, \lambda^{-6} y, \lambda^3 z, \lambda^2 t).$$

If the hypersurface Y would be isomorphic to $\mathbb{C}^3$ then this $\mathbb{C}^\star$-action would not be linearizable, and moreover, the function $f = x + x^2y + z^2 + t^3$ on $\mathbb{C}^4$ would be a counterexample to the Sathaye Conjecture (cf. [57]):

Conjecture 1.2 (Sathaye Conjecture for embeddings of $\mathbb{C}^{n-1}$ into $\mathbb{C}^n$). *If $f^{-1}(0)$ is isomorphic to $\mathbb{C}^{n-1}$, for a polynomial map $f : \mathbb{C}^n \to \mathbb{C}$, then any fiber of f is isomorphic to $\mathbb{C}^{n-1}$.*

However Makar-Limanov showed that

Proposition 1.3. *The hypersurface $Y : x + x^2y + z^2 + t^3 = 0$ is not isomorphic to $\mathbb{C}^3$.*

Let R be a commutative $\mathbb{C}$-algebra. For a derivation $D : R \to R$ we will define R^D as the subring of constants $\{f \in R \mid Df = 0\}$. MAKAR-LIMANOV defined the following invariant of rings:

$$\mathrm{ML}(R) = \bigcap_D R^D$$

where D runs trough all locally nilpotent $\mathbb{C}$-derivations $R \to R$ (see section 2 for the definition and properties of a locally nilpotent derivation). Then

$$\mathrm{ML}(\mathbb{C}[x, y, z]) = \mathbb{C}$$

and MAKAR-LIMANOV proved that

$$\mathrm{ML}\big(\mathbb{C}[x, y, z, t]/(x + x^2y + z^2 + t^3)\big) \neq \mathbb{C}.$$

It follows that the rings $\mathbb{C}[x, y, z]$ and $\mathbb{C}[x, y, z, t]/(x+x^2y+z^2+t^3)$ are not isomorphic, or equivalently, Y is not isomorphic to $\mathbb{C}^3$.

In this chapter, we will give an other proof of the fact that Y is not isomorphic to $\mathbb{C}^3$. We will define a new invariant $C(R)$ as the algebra generated by all R^D where D runs through all non-trivial locally nilpotent derivations. Then we will prove that $C(\mathbb{C}[x, y, z]) = \mathbb{C}[x, y, z]$ and

$$C\big(\mathbb{C}[x, y, z, t]/(x + x^2y + z^2 + t^3)\big) \neq \mathbb{C}[x, y, z, t]/(x + x^2y + z^2 + t^3).$$

And it follows that Y is not isomorphic to $\mathbb{C}^3$.

2. TECHNICAL PREPARATION

In this section rings are always commutative $\mathbb{C}$-algebras.

Filtrations. Suppose that R is a ring with an ascending filtration by subvector spaces $R = \bigcup_{i=-\infty}^{\infty} F^iR$ satisfying $\bigcap_{i=-\infty}^{\infty} F^iR = \{0\}$. The associated graded ring will be denoted by

$$\mathrm{Gr}\, R = \bigoplus_{i=-\infty}^{\infty} F^iR/F^{i-1}R.$$

There is a canonical map $\mathrm{gr} : R \to \mathrm{Gr}\, R$, defined by

$$\mathrm{gr}(f) = f + F^{i-1}R \in F^iR/F^{i-1}R \qquad \text{for } f \in F^iR \setminus F^{i-1}R$$

and $\mathrm{gr}(0) = 0$. Observe that if $f \neq 0$, then $\mathrm{gr}(f) \neq 0$.

Degree functions. A degree function on a ring R is a map $\deg : R \to \mathbb{Z} \cup \{-\infty\}$ satisfying

1. $\deg(f) = -\infty$ if and only if $f = 0$,
2. $\deg(fg) = \deg(f) + \deg(g)$ for all $f, g \in R$,
3. $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$.

A degree function induces a natural filtration, namely

$$F^iR := \{\, f \in R \mid \deg(f) \leq i \,\}$$

and we have $\bigcap_{i=-\infty}^{\infty} F^i R = \{0\}$.

Locally nilpotent derivations. A $\mathbb{C}$-derivation D on R is called *locally nilpotent* if for every $f \in R$ there exists a natural number N such that $D^N f = 0$. There is a 1–1 correspondence between locally nilpotent derivations and regular $\mathcal{G}_a$-actions on $\operatorname{Spec} R$, where $\mathcal{G}_a$ is the additive group $(\mathbb{C}, +)$.

A $\mathcal{G}_a$-action on $\operatorname{Spec} R$ corresponds to an algebra homomorphism

$$\varphi : R \to \mathcal{O}(\mathcal{G}_a) \otimes R \cong R[t]$$

satisfying certain properties. If we define $D : R \to R$ by

$$Df = \left.\frac{d}{dt}\varphi(f)\right|_{t=0}$$

then D is a locally nilpotent derivation on R.

On the other hand, if D is a locally nilpotent $\mathbb{C}$-derivation then

$$f \mapsto \exp(tD)f = \sum_{i=0}^{\infty} \frac{t^i D^i f}{i!}$$

defines an algebra homomorphism $R \to R[t]$ which corresponds to a morphism

$$\mathcal{G}_a \times \operatorname{Spec} R \to \operatorname{Spec} R$$

which is in fact a $\mathcal{G}_a$-action.

If R is a domain and the $\mathcal{G}_a$-action is non-trivial, or equivalently $D \neq 0$, then it is well-known that $\dim R^D = \dim R^{\mathcal{G}_a} = \dim R - 1$.

Lemma 2.1. *Suppose that R is a domain and that D is a locally nilpotent derivation on R. If $f, g \in R \setminus \{0\}$ and $D(fg) = 0$, then $Df = Dg = 0$. It follows that $D(K) = \{0\}$ for any subfield $K \subset R$.*

Proof. If $D(fg) = 0$ then $fg = \exp(tD)(fg) = (\exp(tD)f)(\exp(tD)g)$. Now $\exp(tD)f$ and $\exp(tD)g$ must be constant polynomials in t. So we have $Df = Dg = 0$. For any $c \in \mathbb{C} \setminus \{0\}$ we have $Dc = 0$ because $D(c\frac{1}{c}) = D1 = 0$. □

Lemma 2.2. *If $\mathcal{G}_a$ acts non-trivial on a connected affine variety Z of dimension 1, then Z must be isomorphic to $\mathbb{C}$.*

Proof. Because Z has a non-trivial orbit, it must contain $\mathbb{C}$ as a subvariety. If Z is affine then we must have $Z = \mathbb{C}$. □

Inheritance of locally nilpotent derivations. Suppose that $R = \bigcup_{i=-\infty}^{\infty} F^i R$ is a ring with an ascending filtration such that $\bigcap_{i=-\infty}^{\infty} F^i R = \{0\}$ and D is a (non-trivial) locally nilpotent derivation on R. Suppose moreover, that $DF^iR \subseteq F^{i+k}R$ for some integer k. This condition is satisfied if $\operatorname{Gr} R$ is finitely generated. We may assume that k is as small as possible. Now we define $\operatorname{gr} D : \operatorname{Gr} R \to \operatorname{Gr} R$ by

$$(\operatorname{gr} D) \operatorname{gr} f = Df + F^{i+k-1}R \in F^{i+k}R/F^{i+k-1}R \qquad \text{for } f \in F^iR \setminus F^{i-1}R.$$

It is easy to check that $\operatorname{gr} D$ is a locally nilpotent derivation on $\operatorname{Gr} R$.

Definition. We define the ring invariant $C(R)$ as the algebra generated by all R^D, with D a non-trivial locally nilpotent derivation.

For example, if $R = \mathbb{C}[x, y, z]$ then the kernels of $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ are $\mathbb{C}[y, z]$, $\mathbb{C}[x, z]$ and $\mathbb{C}[x, y]$ respectively. Therefore we get $C(R) = \mathbb{C}[x, y, z] = R$.

3. The proof that $x+x^2y+z^2+t^3=0$ is not isomorphic to $\mathbb{C}^3$

Take $R_1=\mathbb{C}[x,y,z,t]/(x+x^2y+z^2+t^3)$. We can view R_1 as a subring of $\mathbb{C}[z,t,x,x^{-1}]$ if we identify y with $-x^{-2}(x+z^2+t^3)$. We define a degree function on R_1 by

$$\deg_1(f)=-v_x(f) \qquad \text{for all } f\in R_1$$

where v_x is the valuation in x on the ring $\mathbb{C}[z,t,x,x^{-1}]$. For example $\deg_1(z)=0$, $\deg_1(t)=0$, $\deg_1(x)=-1$ and $\deg_1(y)=2$.

Proposition 3.1. *Let D be a non-trivial locally nilpotent derivation on R_1 and $f\in R_1$ a non-zero element. If $Df=0$, then $\deg_1(f)\le 0$.*

Proof of Proposition 1.3. The ring R_1 is not isomorphic to $\mathbb{C}[x,y,z]$ because $C(R_1)\neq R_1$ by Proposition 3.1. □

Proof of Proposition 3.1. Let us assume that there is an element $f_1\in R_1$ with $\deg_1 f_1>0$ and a locally nilpotent derivation $D_1:R_1\to R_1$ such that $D_1f_1=0$. The degree function $\deg_1$ defines a filtration on R_1, and we put $R_2:=\operatorname{Gr}R_1$, $f_2:=\operatorname{gr}f_1$ and $D_2=\operatorname{gr}D_1$. Observe that $D_2f_2=\operatorname{gr}(D_1f_1)=0$.

Lemma 3.2. *If we just write x,y,z,t instead of $\operatorname{gr}x,\operatorname{gr}y,\operatorname{gr}z,\operatorname{gr}t$, then we get*

$$R_2=\mathbb{C}[x,y,z,t]/(x^2y+z^2+t^3).$$

Proof of the lemma. Using the relation $x^2y=-x-z^2-t^3$ we can write every $f\in R_1$ in the form

$$f=a(x,z,t)+yb(y,z,t)+xyc(y,z,t) \qquad \text{for unique polynomials } a,b,c$$

For $\operatorname{gr}f$ there are 3 possibilities:
1. $\operatorname{gr}f=(\operatorname{gr}x)^r p(\operatorname{gr}z,\operatorname{gr}t)$ for some $r\ge 0$, p a polynomial,
2. $\operatorname{gr}f=(\operatorname{gr}y)^r p(\operatorname{gr}z,\operatorname{gr}t)$ for some $r>0$, p a polynomial,
3. $\operatorname{gr}f=\operatorname{gr}x(\operatorname{gr}y)^r p(\operatorname{gr}z,\operatorname{gr}t)$ for some $r>0$, p a polynomial.

So $\operatorname{gr}x,\operatorname{gr}y,\operatorname{gr}z,\operatorname{gr}t$ are generators of the ring R_2. Because

$$x^2y+z^2+t^3=-x\in F^{-1}R_1$$

we get

$$(\operatorname{gr}x)^2\operatorname{gr}y+(\operatorname{gr}z)^2+(\operatorname{gr}t)^3=0$$

More relations between $\operatorname{gr}x$, $\operatorname{gr}y$, $\operatorname{gr}z$ and $\operatorname{gr}t$ would imply that $\dim R_2\le 2$, because $x^2y+z^2+t^3$ is an irreducible polynomial. But we have $\dim R_2\ge 3$, because $\operatorname{gr}x$, $\operatorname{gr}z$ and $\operatorname{gr}t$ are algebraically independent. So we can conclude

$$R_2\cong\mathbb{C}[x,y,z,t]/(x^2y+z^2+t^3).$$

□

Because $\deg_1 f_1 > 0$ there are only two possibilities for $f_2 = \operatorname{gr} f_1 \neq 0$:

1. $f_2 = xy^r p(z,t)$ for some $r > 0$, p a polynomial,
2. $f_2 = y^r p(z,t)$ for some $r > 0$, p a polynomial.

case 1: $f_2 = xy^r p(z,t)$. Because $D_2 f_2 = 0$, we have $D_2 x = D_2 y = 0$ (see Lemma 2.1). We have a morphism:

$$\psi : \operatorname{Spec} R_2 = \{ (x,y,z,t) \in \mathbb{C}^4 \mid x^2y + z^2 + t^3 = 0 \} \to \mathbb{C}^2$$

defined by

$$\psi(x,y,z,t) = (x,y)$$

We look at the $\mathcal{G}_a$-action corresponding to D_2. Now x and y are $\mathcal{G}_a$-invariant functions, therefore $\mathcal{G}_a$ stabilizes the fibers of ψ. The fiber of $(\alpha,\beta) \in \mathbb{C}^2$ is

$$\psi^{-1}(\alpha,\beta) = \{ (\alpha,\beta,z,t) \mid z^2 + t^3 + \alpha^2\beta = 0 \}.$$

For $\alpha,\beta \neq 0$ this is an elliptic curve, hence not isomorphic to $\mathbb{C}$. So the action on these fibers must be trivial by Lemma 2.2. Because these fibers form a dense subset, the $\mathcal{G}_a$-action must be trivial, or equivalently $D_2 = 0$. This is a contradiction. So we must have:

case 2: $f_2 = y^r p(z,t)$. Suppose that p is constant. Because $\dim R_2 = 3$, we have $\dim R_2^{D_2} = 2$. So there is a $g \in R_2^{D_2}$ which is algebraically independent of f_2. We may assume that g is homogeneous with respect to $\deg_2$. For some k we have $\deg_2(f_2^k g) = k\deg_2(f_2) + \deg_2(g) > 0$. If we replace f_2 by $f_2^k g$ then we still have $f_2 \neq 0$, $D_2 f_2 = 0$ and f_2 is of the form $y^{r'}p'(z,t)$. But now $p'(z,t)$ is not constant anymore. So without loss of generality we may assume that p is not constant.

We define a new degree function $\deg_2$ on R_2. We put $\deg_2(x) = 0$, $\deg_2(z) = 3$ and $\deg_2(t) = 2$ ($\deg_2(y) = 6$). We let $R_3 = \operatorname{Gr} R_2$, $f_3 = \operatorname{gr} f_2$ and $D_3 = \operatorname{gr} D_2$. We just write x,y,z,t instead of $\operatorname{gr} x, \operatorname{gr} y, \operatorname{gr} z, \operatorname{gr} t$. Now

$$R_3 = \mathbb{C}[x,y,z,t]/(x^2y + z^2 + t^3)$$

because $\deg_2$ does not only define a filtration, but even grading on R_2, and so R_2 and R_3 are canonically isomorphic. Now f_3 is of the form

$$f_3 = y^r \operatorname{gr} p = y^r z^u t^v \prod_{i=1}^{w} (\alpha_i z^2 + \beta_i t^3) \qquad r > 0,\ u,v,w \geq 0 \text{ and } \alpha_i, \beta_i \in \mathbb{C} \setminus \{0\}$$

Because p is not constant we have either $u > 0$, $v > 0$ or $w > 0$.

case 2a: $u > 0$. Now $D_3 y = D_3 z = 0$. The general fiber of the map

$$(x,y,z,t) \in \{ (x,y,z,t) \mid x^2y + z^2 + t^3 = 0\} \to (y,z)$$

is again an elliptic curve. We get a similar contradiction as in case 1.

case 2b: $v > 0$. Now $D_3 y = D_3 t = 0$. The general fiber of the map

$$(x,y,z,t) \in \{ (x,y,z,t) \mid x^2y + z^2 + t^3 = 0\} \to (y,t)$$

is isomorphic to $\mathbb{C}[x,z]/(x^2+z^2-1)$ which doesn't allow a non-trivial $\mathcal{G}_a$-action. Contradiction.

case 2c: $w>0$. Now $D_3y = D_3(z^2+\gamma t^3)=0$ ($\gamma=\beta_1/\alpha_1$). If $\gamma=1$, then $D_2(x^2y)=0$, so $D_2x=D_2y=0$ and we get a contradiction like in case 1. So assume $\gamma\neq 1$. Let $Z_{\alpha,\beta}$ be the fiber $\psi^{-1}(\alpha,\beta)$ where ψ is defined by

$$\psi:(x,y,z,t)\in\{(x,y,z,t)\mid x^2y+z^2+t^3=0\}\to(y,z^2+\gamma t^3)$$

The functions x,t define a map $Z_{\alpha,\beta}\to\mathbb{C}^2$. The image is one-dimensional and contained in $\{(x,t)\mid x^2\alpha+(1-\gamma)t^3+\beta=0\}$ which is an elliptic curve for $\alpha,\beta\neq 0$. For $\alpha,\beta\neq 0$, $Z_{\alpha,\beta}$ allows a dominant map to an elliptic curve, so it cannot be rational. But then $Z_{\alpha,\beta}$ doesn't allow a $\mathcal{G}_a$-action for $\alpha,\beta\neq 0$ which yields a contradiction.

So the assumption in the beginning that $\deg(f_1)>0$ leads to a contradiction and we conclude $\deg(f_1)\leq 0$. $\square$

Chapter III

Nonlinearizable Holomorphic Group Actions

1. Introduction

The aim of this chapter is to give a negative solution to the following long standing question.

Holomorphic Linearization Problem. *Let $G \hookrightarrow \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ be a complex reductive subgroup of the group $\mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ of holomorphic automorphisms of $\mathbb{C}^n$. Can one conjugate this subgroup by a single automorphism into the general linear group $\mathrm{GL}_n(\mathbb{C}) \subset \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$, i.e., is every action of a complex reductive group on $\mathbb{C}^n$ linearizable?*

The complex algebraic analogue has been studied quite well. For an overview on this subject see the article by Kraft ([42]). The first non-linearizable complex algebraic actions were found by Schwarz in 1989 (see [60]), using non-trivial algebraic G-vector bundles over representations. The problem is still open for abelian groups. For the case $\mathbb{C}^\star$, see the survey by Kraft ([40]). Recently Asanuma ([4]) has constructed non-linearizable multiplicative group actions in the real algebraic category and also in positive characteristic.

Concerning holomorphic linearization several positive results are known: First Suzuki proved that holomorphic $\mathbb{C}^*$-actions on $\mathbb{C}^2$ are linearizable (see [64]). His result was generalized by Jiang (see [29]) who proved using the methods of Kraft and Schwarz (see [44]) that actions of reductive groups on $\mathbb{C}^n$ with one-dimensional quotient are linearizable. Surprisingly the method of constructing counterexamples to the algebraic linearization problem does not work in the holomorphic setting. It follows from the equivariant Oka-principle proved by Heinzner and Kutzschebauch (see [22]) that holomorphic actions arising from equivariant vector bundles over representation spaces are linearizable. Recently Ahern and Rudin proved linearization for finite cyclic groups acting on $\mathbb{C}^2$ by automorphisms lying in a dense subgroup of $\mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^2)$ the group of overshears (see [2]). This result was generalized by Kraft and Kutzschebauch to arbitrary complex reductive groups (see [43]).

In this chapter we construct non-linearizable holomorphic actions on $\mathbb{C}^n$ for all reductive groups. Our main result is (see Theorem 4.10)

Theorem. *For every complex reductive Lie group G (except the trivial group) there exists a natural number N_G such that for all $l \geq N_G$ there exists an effective non-linearizable holomorphic action of G on $\mathbb{C}^l$.*

Our construction is an adaptation of a method of ASANUMA to the holomorphic setting. In [4] he uses exotic algebraic embeddings of $\mathbb{R}$ in $\mathbb{R}^3$ to construct a non-linearizable algebraic $\mathbb{R}^\star$-action on $\mathbb{R}^5$. For a field K of positive characteristic, he constructs a non-linearizable $K^\star$ action on K^4 using an exotic embedding $K \hookrightarrow K^2$. Our construction depends on the existence of non-rectifiable holomorphic embeddings of $\mathbb{C}$ into $\mathbb{C}^n$ for $n \geq 2$. We collected the facts we need about these embeddings for our construction in the second paragraph. In the third paragraph we define the Rees spaces, a holomorphic analogue of the Rees algebra, and prove some of their properties. The main idea of the construction is explained in the first part of paragraph 4, where we give counterexamples for actions of $\mathbb{C}^*$. In the remaining part of the chapter we give similar constructions and use some representation theory to obtain counterexamples for all reductive groups.

I wouldn't have found the results of this chapter without the many discussions with FRANK KUTZSCHEBAUCH, who explained the theory of non-rectifiable embeddings to me. He also helped me to carry out the ideas of ASANUMA in the holomorphic category to obtain non-linearizable actions for $\mathbb{C}^\star$. The generalization to arbitrary groups is based on my ideas.

2. NON-RECTIFIABLE EMBEDDINGS

In this section we explain how one constructs holomorphic embeddings of $\mathbb{C}^k$ into $\mathbb{C}^n$ $(0 < k < n)$ which are not equivalent to the standard embedding. Such embeddings were found by FORSTNERIC, GLOBEVNIK and ROSAY in [16]. Since we need some special version of their result (see Corollary 2.4) we recall the main ideas.

A holomorphic map $F : \mathbb{C}^n \to \mathbb{C}^n$ is called nondegenerate if $JF(p) \neq 0$ for some $p \in \mathbb{C}^n$, where JF is the determinant of the Jacobi matrix.

Theorem 2.1 (Rudin, Rosay). *If $n > 1$ then there exists a discrete subset $E \subset \mathbb{C}^n$ such that all nondegenerate $F : \mathbb{C}^n \to \mathbb{C}^n$ satisfy $F(\mathbb{C}^n) \cap E \neq \emptyset$ (cf. [56]).*

Such discrete subsets $E \subset \mathbb{C}^n$ are called *unavoidable.*

Theorem 2.2 (Forstneric, Globevnik, Rosay). *For any discrete subset $E \subset \mathbb{C}^2$ there exists a proper holomorphic embedding $\varphi : \mathbb{C} \hookrightarrow \mathbb{C}^2$ such that $\varphi(\mathbb{C}) \supset E$ (cf. [16]).*

Definition. A proper holomorphic embedding $\varphi : \mathbb{C}^k \to \mathbb{C}^n$ is called *rectifiable* if there exists a $\beta \in \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ such that

$$\beta \circ \varphi(z_1, \ldots, z_k) = (z_1, \ldots, z_k, 0, \ldots, 0).$$

REMARK 2.3. Every automorphism of $\mathbb{C}^k \times \{0\} \subset \mathbb{C}^n$ extends to an automorphism of $\mathbb{C}^n$. It follows that a proper holomorphic embedding $\varphi : \mathbb{C}^k \to \mathbb{C}^n$ is rectifiable if and only if there exists a $\beta \in \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$ such that $\beta(\varphi(\mathbb{C}^k)) = \mathbb{C}^k \times \{0\} \subset \mathbb{C}^n$.

Take an unavoidable subset $E \subset \mathbb{C}^2$ and let $\varphi : \mathbb{C} \to \mathbb{C}^2$ be a proper holomorphic embedding such that $\varphi(\mathbb{C}) \supset E$. Define $\varphi_k : \mathbb{C} \times \mathbb{C}^k \to \mathbb{C}^2 \times \mathbb{C}^k$ by $\varphi_k(z, y) = (\varphi(z), y)$.

Corollary 2.4. *The universal covering of $\mathbb{C}^{2+k} \setminus \varphi_k(\mathbb{C}^{1+k}) = (\mathbb{C}^2 \setminus \varphi(\mathbb{C})) \times \mathbb{C}^k$ is not biholomorphic to $\mathbb{C}^{2+k}$. So the embedding φ_k is not rectifiable.*

Proof. Suppose $\pi : \mathbb{C}^{k+2} \to (\mathbb{C}^2 \setminus \varphi(\mathbb{C})) \times \mathbb{C}^k$ is the universal covering and let $p : (\mathbb{C}^2 \setminus \varphi(\mathbb{C})) \times \mathbb{C}^k \to \mathbb{C}^2 \setminus \varphi(\mathbb{C})$ be the projection onto the first factor. The composition $p \circ \pi : \mathbb{C}^{k+2} \to \mathbb{C}^2 \setminus \varphi(\mathbb{C})$ is a submersion. The restriction of $p \circ \pi$ to a generic linear two-dimensional subspace is nondegenerate. This contradicts the unavoidability of $E \subset \varphi(\mathbb{C})$. □

The following lemma shows that every embedding of $\mathbb{C}^k$ into $\mathbb{C}^n$ whose image lies in a linear subspace of codimension at least k is rectifiable, so adding some additional dimensions to $\mathbb{C}^n$ one can rectify every embedding inside the higher-dimensional space. We need this fact at the end of paragraph 3 to show that the product of some Rees spaces with $\mathbb{C}^l$ $(l \gg 0)$ are affine spaces.

Lemma 2.5. *Let X be a Stein space. If $\varphi_1 : X \to \mathbb{C}^n$ and $\varphi_2 : X \to \mathbb{C}^m$ are two proper holomorphic embeddings, then there exists a $\beta \in \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n \times \mathbb{C}^m)$ such that $\beta \circ (\varphi_1 \times 0) = 0 \times \varphi_2$. (compare with Lemma 3.2 in [4]).*

Proof. By a Corollary of Cartan's Theorem B the holomorphic map $\varphi_1 : X \to \mathbb{C}^n$ can be extended to a holomorphic map μ_1 from $\mathbb{C}^m \supseteq \varphi_2(X)$ to $\mathbb{C}^n$ (so $\mu_1 \circ \varphi_2 = \varphi_1$). Likewise there exists a holomorphic map $\mu_2 : \mathbb{C}^n \to \mathbb{C}^m$ such that $\mu_2 \circ \varphi_1 = \varphi_2$. Define $\beta_1, \beta_2 \in \mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n \times \mathbb{C}^m)$ by $\beta_1(x, y) = (x, y + \mu_2(x))$ and $\beta_2(x, y) = (x + \mu_1(y), y)$. Now we have

$$\beta_1 \circ (\varphi_1 \times 0) = \varphi_1 \times \varphi_2 = \beta_2 \circ (0 \times \varphi_2).$$

So take $\beta = \beta_2^{-1} \circ \beta_1$. □

3. Rees spaces

In this section we introduce the Rees spaces. Important for the next paragraph are especially Example 3.5 and Corollary 3.7.

Let X, Y and Z be Stein spaces such that $X \subset Y$ and let $f \in \mathcal{O}(Z)$ be a holomorphic function on Z. Suppose that there are finitely many holomorphic functions $h_1, h_2, \ldots, h_r$ on Y generating the ideal $I_X(Y)$ of X. This is for instance true if X and Y are manifolds (see [15], Satz 5.5). Consider the map:

$$\psi : Y \times (Z \setminus \{f = 0\}) \to Y \times Z \times \mathbb{C}^r$$
$$\psi(y, z) = \left(y, z, \frac{h_1(y)}{f(z)}, \frac{h_2(y)}{f(z)}, \ldots, \frac{h_r(y)}{f(z)}\right).$$

Definition. We define the *Rees space* $\mathcal{R}(X, Y, Z, f)$ as $\overline{\mathrm{Im}(\psi)}$, the holomorphic closure of the image of ψ within $Y \times Z \times \mathbb{C}^r$.

Remark 3.1. The holomorphic closure of $\mathrm{Im}(\psi)$ within $Y \times Z \times \mathbb{C}^r$ equals the closure in the $\mathbb{C}$-topology.

Remark 3.2. The definition of the Rees space doesn't depend on the choice of generators $h_1, h_2, \ldots, h_r$. Suppose that $g_1, g_2, \ldots, g_s$ also generate $I_X(Y)$, and let ψ' be the corresponding map $Y \times (Z \setminus \{f = 0\}) \to Y \times Z \times \mathbb{C}^s$. It is easy to see that there exist holomorphic maps $\varphi : Y \times Z \times \mathbb{C}^r \to Y \times Z \times \mathbb{C}^s$ and $\varphi' : Y \times Z \times \mathbb{C}^s \to Y \times Z \times \mathbb{C}^r$ such that $\psi' = \varphi \circ \psi$ and $\psi = \varphi' \circ \psi'$. Now $\varphi' \circ \varphi = \mathrm{id}$ on $\mathrm{Im}(\psi)$, therefore $\varphi' \circ \varphi = \mathrm{id}$ on $\overline{\mathrm{Im}(\psi)}$ and likewise $\varphi \circ \varphi' = \mathrm{id}$ on $\overline{\mathrm{Im}(\psi')}$. So the restriction of φ defines a biholomorphic map $\overline{\mathrm{Im}(\psi)} \xrightarrow{\cong} \overline{\mathrm{Im}(\psi')}$.

REMARK 3.3. If we take $Z := \mathbb{C}$ and $f := z$, then the Rees space is just the holomorphic analogue of the Rees algebra.

We will use the symbol $\cong$ to denote biholomorphic.

Lemma 3.4. *Rees spaces have the following properties:*

1. $\mathcal{R}(X, Y, Z, f) \cong \mathcal{R}(\varphi(X), Y, Z, f)$ *for* $\varphi \in \mathrm{Aut}_{\mathrm{hol}}(Y)$,
2. $\mathcal{R}(X \times W, Y \times W, Z, f) \cong \mathcal{R}(X, Y, Z, f) \times W$ *for any Stein space* W,
3. $\mathcal{R}(X \times \{0\}, Y \times \mathbb{C}^m, Z, f) \cong \mathcal{R}(X, Y, Z, f) \times \mathbb{C}^m$.

PROOF. The first two properties are trivial. We show the third property. Suppose $I_X(Y)$ is generated by $h_1, h_2, \ldots, h_r \in \mathcal{O}(Y)$. Then $I_{X\times\{0\}}(Y \times \mathbb{C}^m)$ is generated by $h_1, \ldots, h_r, u_1, \ldots, u_m$ where the u_i are the coordinate functions of $\mathbb{C}^m$. Consider the map

$$\psi : Y \times \mathbb{C}^m \times (Z \setminus \{f = 0\}) \to Y \times \mathbb{C}^m \times Z \times \mathbb{C}^r \times \mathbb{C}^m$$
$$\psi(y, u, z) = (y, u, z, f(z)^{-1} \cdot h, f(z)^{-1} \cdot u),$$

where $h = (h_1, \ldots, h_r)$ and $u = (u_1, \ldots, u_m)$. Define $\varphi \in \mathrm{Aut}_{\mathrm{hol}}(Y \times \mathbb{C}^m \times Z \times \mathbb{C}^r \times \mathbb{C}^m)$ by $\varphi(y, u, z, v, w) = (y, u - f(z).w, z, v, w)$. Now we have

$$\varphi \circ \psi(y, u, z) = (y, 0, z, f(z)^{-1}.h, f(z)^{-1}.u),$$

and we get

$$\mathcal{R}(X \times \{0\}, Y \times \mathbb{C}^m, Z, f) = \overline{\mathrm{Im}(\psi)} \cong \overline{\mathrm{Im}(\varphi \circ \psi)} \cong \mathcal{R}(X, Y, Z, f) \times \mathbb{C}^m.$$

□

EXAMPLE 3.5. Suppose $X \subset Y$ is given by $h = 0$ with $h \in \mathcal{O}(Y)$ $(I_X(Y) = \mathcal{O}(Y)h)$. Let $\psi : Y \times (Z \setminus \{f = 0\}) \to Y \times Z \times \mathbb{C}$ given by $\psi(y, z) = (y, z, h(y)/f(z))$. Then we get

$$\mathcal{R}(X, Y, Z, f) = \overline{\mathrm{Im}(\psi)} = \{(y, z, w) \in Y \times Z \times \mathbb{C} \mid f(z)w = h(y)\}.$$

Lemma 3.6. *If* $\varphi_1 : X \to \mathbb{C}^n$ *and* $\varphi_2 : X \to \mathbb{C}^m$ *are proper holomorphic embeddings, then*

$$\mathcal{R}(\varphi_1(X), \mathbb{C}^n, Z, f) \times \mathbb{C}^m \cong \mathcal{R}(\varphi_2(X), \mathbb{C}^m, Z, f) \times \mathbb{C}^n$$

(compare with [4], Corollary 4.2).

PROOF. From Lemma 2.5 and Lemma 3.4 it follows that

$$\mathcal{R}(\varphi_1(X), \mathbb{C}^n, Z, f) \times \mathbb{C}^m \cong \mathcal{R}(\varphi_1(X) \times \{0\}, \mathbb{C}^n \times \mathbb{C}^m, Z, f) \cong$$
$$\cong \mathcal{R}(\{0\} \times \varphi_2(X), \mathbb{C}^n \times \mathbb{C}^m, Z, f) \cong \mathcal{R}(\varphi_2(X), \mathbb{C}^m, Z, f) \times \mathbb{C}^n.$$

□

Corollary 3.7. *For an embedding* $\varphi : \mathbb{C}^k \to \mathbb{C}^n$ *and* $f \in \mathcal{O}(\mathbb{C}^l)$ *we have*

$$\mathcal{R}(\varphi(\mathbb{C}^k), \mathbb{C}^n, \mathbb{C}^l, f) \times \mathbb{C}^k \cong \mathcal{R}(\mathbb{C}^k, \mathbb{C}^k, \mathbb{C}^l, f) \times \mathbb{C}^n \cong \mathbb{C}^{k+l+n}.$$

4. Non-linearizable group actions

The goal of this section is to find for every non-trivial reductive group a non-linearizable action on $\mathbb{C}^n$ for some n. As usually we denote by X^G the fixed point set of an action of G on X. Several times we will use the notion of categorical quotient $\pi_X : X \to X/\!\!/G$ for a holomorphic action of a reductive group G on a Stein space X. Recall that this is the quotient of X with respect to the equivalence relation which is given by the invariant holomorphic functions and that $X/\!\!/G$ carries a complex structure such that π_X is a holomorphic map. Since G-invariant holomorphic functions on a closed G-invariant subspace Y of a holomorphic Stein G-space X extend to G-invariant holomorphic functions on X, the categorical quotient $Y \to Y/\!\!/G$ is just given by restricting π_X to Y. Beside some easy examples of quotients for linear actions this is the only fact we use. So for further details on categorical quotients we refer to the papers of Snow ([63]) and Heinzner ([21]).

Let $\varphi = \varphi_0 : \mathbb{C} \to \mathbb{C}^2$ be a proper holomorphic embedding like in Corollary 2.4. By the solution of the second Cousin problem we find a function $h \in \mathcal{O}(\mathbb{C}^2)$ which vanishes precisely on $\varphi(\mathbb{C})$ of order 1.

Non-linearizable $\mathbb{C}^\star$-actions

Consider the Rees space $X = \mathcal{R}(\varphi(\mathbb{C}), \mathbb{C}^2, \mathbb{C}, z)$. Like in Example 3.5 we have

$$X = \{(x, y, z, w) \in \mathbb{C}^4 \mid h(x, y) = zw\}.$$

By Corollary 3.7 there exists a biholomorphism $\alpha_k : X \times \mathbb{C}^k \xrightarrow{\cong} \mathbb{C}^{k+3}$ for all $k \geq 1$. Now $X \times \mathbb{C}^k \subset \mathbb{C}^4 \times \mathbb{C}^k$ is stable under the linear $\mathbb{C}^\star$-action on $\mathbb{C}^4 \times \mathbb{C}^k$ given by

$$\mathbb{C}^\star \times \mathbb{C}^4 \times \mathbb{C}^k \to \mathbb{C}^4 \times \mathbb{C}^k$$

$$\lambda \cdot (x, y, z, w, t_1, \dots, t_k) = (x, y, \lambda z, \lambda^{-1} w, t_1, \dots, t_k).$$

The restriction of this action to $X \times \mathbb{C}^k$ induces via α_k a holomorphic $\mathbb{C}^\star$-action $\sigma_k : \mathbb{C}^\star \times \mathbb{C}^{k+3} \to \mathbb{C}^{k+3}$.

Proposition 4.1. *The action σ_k is not linearizable ($k \geq 1$). So for all $l \geq 4$ there exists a non-linearizable $\mathbb{C}^\star$ action on $\mathbb{C}^l$.*

Proof. Suppose $\alpha_k : X \times \mathbb{C}^k \xrightarrow{\cong} \mathbb{C}^{3+k}$ is a biholomorphic $\mathbb{C}^\star$-equivariant map, where $\mathbb{C}^\star$ acts linearly on $\mathbb{C}^{3+k}$. This representation of $\mathbb{C}^\star$ on $\mathbb{C}^{k+3}$ must be isomorphic to the representation of $\mathbb{C}^\star$ on the tangent space of some fixed point of $X \times \mathbb{C}^k$. With respect to some coordinates, this action is given by

$$\lambda \cdot (z, w, u_1, \dots, u_{k+1}) = (\lambda z, \lambda^{-1} w, u_1, \dots, u_{k+1}).$$

The categorical quotient of $X \times \mathbb{C}^k$ is given by $\pi_1 : X \times \mathbb{C}^k \to \mathbb{C}^{2+k}$,

$$\pi_1(x, y, z, w, t_1, \dots, t_k) = (x, y, t_1, \dots, t_k)$$

and the categorical quotient of $\mathbb{C}^{3+k}$ is given by $\pi_2 : \mathbb{C}^{3+k} \to \mathbb{C}^{2+k}$,

$$\pi_2(z, w, u_1, \dots, u_{k+1}) = (zw, u_1, \dots, u_{k+1}).$$

The fixed point set $(X \times \mathbb{C}^k)^{\mathbb{C}^*}$ is

$$\{(x, y, z, w, t_1, \dots, t_k) \in X \times \mathbb{C}^k \mid h(x, y) = z = w = 0\}.$$

Its image under π_1 is $\varphi(\mathbb{C}) \times \mathbb{C}^k \subset \mathbb{C}^2 \times \mathbb{C}^k$. On the other hand $(\mathbb{C}^{3+k})^{\mathbb{C}^*}$ is

$$\{(z, w, u_1, \dots, u_{k+1}) \in \mathbb{C}^{3+k} \mid z = w = 0\}$$

and its image under π_2 is $\{0\} \times \mathbb{C}^{1+k} \subset \mathbb{C}^{2+k}$. Now $\alpha_k : X \times \mathbb{C}^k \xrightarrow{\cong} \mathbb{C}^{3+k}$ induces a biholomorphism $\gamma : \mathbb{C}^{2+k} \xrightarrow{\cong} \mathbb{C}^{2+k}$ such that $\gamma(\varphi(\mathbb{C}) \times \mathbb{C}^k) = \{0\} \times \mathbb{C}^{k+1}$. This contradicts Corollary 2.4. □

Non-linearizable actions for finite cyclic groups

Suppose $G \cong \mathbb{Z}/n\mathbb{Z}$ $(n \geq 2)$. Define a holomorphic function $f \in \mathcal{O}(\mathbb{C}^{n-1})$ by $f(z_1, z_2, \dots, z_{n-1}) = z_1 z_2 \cdots z_{n-1}$. Consider the Rees space $X := \mathcal{R}(\varphi(\mathbb{C}), \mathbb{C}^2, \mathbb{C}^{n-1}, f)$ which is given by (see Example 3.5)

$$X = \{(x, y, z_1, z_2, \dots, z_n) \in \mathbb{C}^{2+n} \mid h(x, y) = z_1 z_2 \cdots z_n\}.$$

From Corollary 3.7 we get $X \times \mathbb{C}^k \cong \mathbb{C}^{1+n+k}$ for $k \geq 1$. Let G act on $\mathbb{C}^2 \times \mathbb{C}^n$ by

$$g \cdot (x, y, z_1, z_2, \dots, z_n) = (x, y, z_2, z_3, \dots, z_n, z_1)$$

where g is a generator of G. This induces by restriction an action of G on X. If we let G act trivially on $\mathbb{C}^k$ then we get an action of G on $\mathbb{C}^{1+n+k} \cong X \times \mathbb{C}^k$ $(k \geq 1)$ which we denote by σ_k. The fixed point set of G on $X \times \mathbb{C}^k$ is biholomorphic to $Y_n \times \mathbb{C}^k$ where

$$Y_n = \{(x, y, z) \mid h(x, y) = z^n\}.$$

We define a G action on Y_n by

$$g \cdot (x, y, z) = (x, y, \zeta z)$$

where ζ is a primitive n-th root of unity. If we let G act trivially on $\mathbb{C}^k$ then we get an action of G on $Y_n \times \mathbb{C}^k$ which we will denote by τ_k.

Proposition 4.2. *For $k \geq 1$ at least one of the following statements is true:*

1. *The action σ_k on $\mathbb{C}^{n+k+1}$ is not linearizable.*
2. *$Y_n \times \mathbb{C}^l \cong \mathbb{C}^{2+l}$ and the action τ_l of G on $Y_n \times \mathbb{C}^l$ is not linearizable for all $l \geq k$.*

Proof. If the action σ_k is linearizable, then its fixed point set $Y_n \times \mathbb{C}^k$ must be a vector space, so $Y_n \times \mathbb{C}^l \cong \mathbb{C}^{2+l}$ for all $l \geq k$. The fixed point set of G on $Y_n \times \mathbb{C}^l$ is isomorphic to $W \times \mathbb{C}^l$ where W is given by

$$W = \{(x, y, z) \in \mathbb{C}^3 \mid h(x, y) = z = 0\}$$

If τ_l is linearizable, then $Y_n \times \mathbb{C}^l \setminus W \times \mathbb{C}^l = (Y_n \setminus W) \times \mathbb{C}^l$ is biholomorphic to $\mathbb{C}^\star \times \mathbb{C}^{l+1}$. But this contradicts Lemma 4.3 below. □

Lemma 4.3. *For $k \geq 0$ the space $(Y_n \setminus W) \times \mathbb{C}^k$ is not biholomorphic to $\mathbb{C}^\star \times \mathbb{C}^{1+k}$.*

Proof. The map $(x, y, z) \in (Y_n \setminus W) \to (x, y) \in \mathbb{C}^2 \setminus \varphi(\mathbb{C})$ is an n-sheeted unramified covering of $\mathbb{C}^2 \setminus \varphi(\mathbb{C})$. So $(Y_n \setminus W) \times \mathbb{C}^k$ and $(\mathbb{C}^2 \setminus \varphi(\mathbb{C})) \times \mathbb{C}^k$ have the same universal covering. But by Corollary 2.4 this universal covering is not biholomorphic to $\mathbb{C}^{2+k}$. □

Corollary 4.4. *For $l \geq n + 2$ there exists a non-linearizable action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{C}^l$.*

Nonlinearizable actions for simple non-abelian groups

Define

$$X_n := \{(x, y, z_1, \ldots, z_n, w_1, \ldots, w_n) \in \mathbb{C}^{2n+2} \mid h(x, y) = z_1 w_1 + \ldots + z_n w_n\}.$$

Lemma 4.5. *The space $X_n \times \mathbb{C}^{3\cdot 2^n - 2n - 3}$ is biholomorphic to $\mathbb{C}^{3\cdot 2^n - 2}$.*

Proof. We prove the lemma by induction. For $n = 0$ this is trivial. Suppose that $X_n \times \mathbb{C}^{3\cdot 2^n - 2n - 3} \cong \mathbb{C}^{3\cdot 2^n - 2}$. Observe that $X_{n+1} \cong \mathcal{R}(X_n, \mathbb{C}^{2n+2}, \mathbb{C}, z)$ (see Example 3.5). By Corollary 3.7 we have

$$X_{n+1} \times \mathbb{C}^{3\cdot 2^{n+1} - 2(n+1) - 3} \cong \mathcal{R}(X_n, \mathbb{C}^{2n+2}, \mathbb{C}, z) \times \mathbb{C}^{3\cdot 2^n - 2n - 3} \times \mathbb{C}^{3\cdot 2^n - 2} \cong$$
$$\mathcal{R}(X_n \times \mathbb{C}^{3\cdot 2^n - 2n - 3}, \mathbb{C}^{2n+2} \times \mathbb{C}^{3\cdot 2^n - 2n - 3}, \mathbb{C}, z) \times \mathbb{C}^{3\cdot 2^n - 2} \cong \mathbb{C}^{3\cdot 2^{n+1} - 2}.$$

□

Suppose that there exists a subgroup H of G, an n-dimensional representation V of G and a homogeneous G-invariant polynomial a on V of degree 2 such that:

- $V^G = \{0\}$,
- $\dim V^H = 1$ and **(*)**
- a is not identical to 0 on V^H.

Then we define

$$U_1 := \{(x, y, v) \in \mathbb{C}^2 \times V \mid h(x, y) = a(v)\}$$
$$U_2 := \{(x, y, v, w) \in \mathbb{C}^2 \times V \times V \mid h(x, y) = a(v) + a(w)\}.$$

The spaces U_1 and U_2 carry G-actions defined by

$$g \cdot (x, y, v) = (x, y, g \cdot v) \qquad (x, y, v) \in U_1 \subset \mathbb{C}^2 \times V, g \in G$$
$$g \cdot (x, y, v, w) = (x, y, v, g \cdot w) \qquad (x, y, v, w) \in U_2 \subset \mathbb{C}^2 \times V \times V, g \in G.$$

Proposition 4.6. *For $k \geq 3 \cdot 2^n - 2n - 3$ we have $U_2 \times \mathbb{C}^k \cong \mathbb{C}^{k+2n+1}$ and at least one of the following statements is true:*

1. *The action of G on $U_2 \times \mathbb{C}^k$ is not linearizable.*
2. *$U_1 \times \mathbb{C}^l \cong \mathbb{C}^{l+n+1}$ for $l \geq k$ and the action of G on $U_1 \times \mathbb{C}^l$ is not linearizable.*

Proof. After a change of coordinates we have

$$a(z) = z_1^2 + z_2^2 + \ldots + z_r^2$$

for $z = (z_1, z_2, \ldots, z_n)$ with respect to some coordinates on V. So

$$a(z) + a(w) = z_1^2 + z_2^2 + \ldots + z_r^2 + w_1^2 + w_2^2 + \ldots + w_r^2$$

where $w = (w_1, w_2, \ldots, w_n)$. Again after a linear change of coordinates in $V \times V$ we get

$$a(z) + a(w) = z'_1 w'_1 + z'_2 w'_2 + \ldots + z'_r w'_r.$$

So $U_2 \cong X_r \times \mathbb{C}^{2n-2r}$ and it follows from Corollary 3.7 that $U_2 \times \mathbb{C}^k \cong \mathbb{C}^{k+2n+1}$ for all $k \geq 3 \cdot 2^n - 2n - 3$. The fixed point set of G on $U_2 \times \mathbb{C}^k$ is $U_1 \times \mathbb{C}^k$. So if the action of G on $U_2 \times \mathbb{C}^k$ is linearizable then $U_1 \times \mathbb{C}^l \cong \mathbb{C}^{l+n+1}$ for all $l \geq k$. Take $p \in V$ such that $V^H = \mathbb{C} \cdot p$. Then we have

$$(U_1 \times \mathbb{C}^l)^H = \{(x, y, z) \in \mathbb{C}^2 \times \mathbb{C} \cdot p \mid h(x, y) = a(z)\} \times \mathbb{C}^l$$

Because $a(p) \neq 0$, this is biholomorphic to

$$\{(x, y, z) \in \mathbb{C}^3 \mid h(x, y) = z^2\} \times \mathbb{C}^l = Y_2 \times \mathbb{C}^l.$$

On the other hand $(U_1 \times \mathbb{C}^l)^G$ is the subset given by

$$\{(x, y, z) \in \mathbb{C}^3 \mid h(x, y) = z = 0\} \times \mathbb{C}^l = W \times \mathbb{C}^l.$$

The space $(U_1 \times \mathbb{C}^l)^H \setminus (U_1 \times \mathbb{C}^l)^G = (Y_2 \setminus W) \times \mathbb{C}^l$ is not biholomorphic to $\mathbb{C}^\star \times \mathbb{C}^{l+1}$ by Lemma 4.3. Therefore the G-action on $U_1 \times \mathbb{C}^l$ cannot be linearizable. □

In the remaining part of this paragraph we show in a case by case study that for all simple non-abelian groups the setting (*) can be achieved thus by Proposition 4.6 constructing non-linearizable actions.

First suppose G is a positive dimensional simple Lie group. Let $V := \mathfrak{g}$ be the Lie algebra of G. Consider the adjoint action of G on $\mathfrak{g}$. Fix a maximal torus $T \subset G$. The fixed point set of T is $\mathfrak{t}$, the Lie algebra of T. Now the Weyl group $\mathcal{W} = N_G(T)/T$ acts on $\mathfrak{t}$. Suppose that $s_1, s_2, \ldots, s_r$ are the simple generating reflections of W corresponding with the simple roots $\alpha_1, \ldots, \alpha_r$. The Killing form $\langle \cdot, \cdot \rangle$ given by $(a, b) := \mathrm{Trace}(\mathrm{ad}(a)\mathrm{ad}(b))$ is a G-invariant bilinear form on $\mathfrak{g}$. The restriction to $\mathfrak{t}$ is nondegenerate, so we can identify $\mathfrak{t}$ with its dual. Let $\mathcal{W}'$ be the subgroup of W generated by $s_1, s_2, \ldots, s_{r-1}$. The fixed point set of W' is one dimensional and spanned by the fundamental weight λ_r. Let H be the group containing T such that $H/T = \mathcal{W}'$. If we define $a(v) := (v, v)$ then the restriction of a to V^H is not identically 0 because $a(\lambda_r) = (\lambda_r, \lambda_r) > 0$. So by Proposition 4.6 we obtain:

Corollary 4.7. *If G is a positive dimensional simple Lie group then for all $l \geq 3 \cdot 2^{\dim G} - 2$ there exist a non-linearizable action of G on $\mathbb{C}^l$.*

Second we consider the case where G is a non-abelian finite simple group. The proof of the following lemma was shown to us by Jürgen Müller:

Lemma 4.8. *For any non-abelian finite simple group G, there exist a subgroup H and a real representation $V_\mathbb{R}$ such that $V_\mathbb{R}^G = \{0\}$ and $V_\mathbb{R}^H$ is one-dimensional.*

Proof. We use the classification of finite simple groups.

Alternating groups: For $G = A_n$ $(n \geq 5)$, take

$$V_\mathbb{R} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + x_2 + \ldots + x_n = 0\}$$

and let G act on it by permuting the coordinates. Take $H = A_{n-1}$, the alternating subgroup which permutes only the first $n - 1$ coordinates. It is clear that $V_\mathbb{R}^H$ is one-dimensional, spanned by the vector $(1, 1, \ldots, 1, 1 - n)$.

For the groups of Lie type and the sporadic ones we will look for a non-trivial irreducible real character χ such that $\langle \chi, (1_H)^G \rangle = 1$, where $(1_H)^G$ is the character of G induced by the trivial character 1_H of H. Now $(1_H)^G$ is the character of the real representation $\mathbb{R}[G/H]$. Let $V_{\mathbb{R}}$ be the isotypic component of χ in $\mathbb{R}[G/H]$. The multiplicity of χ in $V_{\mathbb{R}}$ is $\langle \chi, (1_H)^G \rangle = 1$. So $V_{\mathbb{R}}$ is a real representation with character χ. The dimension of $V_{\mathbb{R}}^H$ is $\langle \chi_H, 1_H \rangle$ which is equal to $\langle \chi, (1_H)^G \rangle = 1$ by Frobenius reciprocity.

Finite simple groups of Lie type: Take $H := B$, the Borel subgroup. Let I be the set of simple reflections. For every subset $J \subseteq I$ we have a parabolic subgroup P_J. Let $(1_{P_J})^G$ be the character of G induced by the trivial character 1_{P_J} of P_J. Define a function St on G by

$$\mathrm{St} = \sum_{J \subseteq I} (-1)^{|J|} (1_{P_J})^G$$

This turns out to be an irreducible character and it has the following properties (see Chapter 6 of [8]):

- $\langle \mathrm{St}, (1_B)^G \rangle = 1$,
- $\mathrm{St}(1) = |U|$ where U is the maximal unipotent subgroup.

Now St is clearly a real character, and it belongs to a real representation $V_{\mathbb{R}}$ of dimension $|U|$ satisfying $\dim V_{\mathbb{R}}^H = 1$.

Sporadic groups: If $(1_H)^G = \sum_{i=1}^r a_i \chi_i$ where $a_1, a_2, \ldots, a_r \in \mathbb{Z}$ and $\chi_1, \chi_2, \ldots, \chi_r$ different irreducible characters, then $a_i = \langle \chi_i, (1_H)^G \rangle$. For most of the sporadic groups and some of its large subgroups the decomposition of $(1_H)^G$ can be found in the *Atlas of Finite Groups* (see [9]). The table below shows the possibilities for G, H and χ as they can be found in the Atlas. The notation of the Atlas is used. For example, if $G = J_3$ (the third Janko group) then 323a stands for the first 323-dimensional irreducible representation of J_3 as listed in the character table of J_3 in the Atlas.

group G	subgroup H	character
M_{11}	M_{10}	10a
M_{12}	M_{11}	11a
M_{22}	$L_3(4)$	21a
M_{23}	M_{22}	22a
M_{24}	M_{23}	23a
J_1	$L_2(11)$	56a
J_2	$U_3(3)$	36a
J_3	$L_2(16):2$	323a
Suz	$G_2(4)$	780a
HS	M_{22}	22a
McL	$U_4(3)$	22a
Co_3	$McL:2$	275a
Co_2	$U_6(2):2$	275a
He	$S_4(4):2$	680a
Fi_{22}	$2\cdot U_6(2)$	429a
Fi_{23}	$2\cdot Fi_{22}$	782a
Fi'_{24}	Fi_{23}	57477a
$O'N$	$L_3(7):2$	10944a
Ly	$G_2(5)$	45694a
Ru	${}^2F_4(2)$	783a

For the other sporadic groups the computer algebra system *GAP* can be used to find:

group G	subgroup H	character
J_4	$2^{11}:M_{24}$	889111a
Co_1	Co_2	299a
HN	A_{12}	133a
Th	${}^3D_4(2):3$	4123a
B	Th	96255a
M	$2\cdot B$	196883a

For all sporadic groups we took for H the maximal subgroup of G, except in the cases $G = B$ and $G = M$ where the maximal subgroups aren't classified yet. $\square$

Now we realize the setting **(*)**. Take H and $V_{\mathbb{R}}$ as in Lemma 4.8. Put $V := V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Let $p \in V_{\mathbb{R}}^H \setminus \{0\}$. Choose a real linear function $b \in V_{\mathbb{R}}^\star$ such that $b(p) \neq 0$. Define the G-invariant polynomial $a = \sum_{g\in G}(g\cdot b)^2$ on V. Now $V^G = \{0\}$, $V^H = \mathbb{C}\cdot p$ and $a(p) > 0$. From Proposition 4.6 we obtain:

Corollary 4.9. *If G is a finite simple group, then there exists a non-linearizable action of G on $\mathbb{C}^l$ for all $l \geq 3\cdot 2^d - 2$ where*

- *$d = n-1$ if $G = A_n$,*
- *$d = \mathrm{St}(1) = |U|$ if G is of Lie type,*
- *d is like in the tables of the proof of Lemma 4.8 if G is sporadic.*

Effective non-linearizable actions for all reductive groups

Theorem 4.10. *For every complex reductive Lie group G (except the trivial group) there exists a natural number N_G such that for all $l \geq N_G$ there exists an effective non-linearizable holomorphic action of G on $\mathbb{C}^l$.*

Proof. Let H be the maximal closed normal subgroup of G. For all $l \geq N_{G/H}$ there exists an effective non-linearizable action of G/H on $\mathbb{C}^l$ by Proposition 4.1, Corollary 4.4, Corollary 4.7 and Corollary 4.9. Take a G-representation V on which H acts effectively. Let $W = V/V^H$. Now H acts on W effectively and $W^H = \{0\}$. The action of G on $\mathbb{C}^l \times W$ is effective. If this action were linearizable, then the action of G on $(\mathbb{C}^l \times W)^H = \mathbb{C}^l \times \{0\}$ would be linearizable, but this is not the case. $\square$

Glossary of Notations

Chapter I

k	base field, 1
G	linear algebraic group, 1
ρ	representation of G, 1
V	n-dimensional G-module, 1
$\mathcal{O}(V)$	coordinate ring of V, 1
$\mathcal{O}(V)^G$	invariant ring, 1
$\mathrm{char}(k)$	characteristic of the field k, 1
$\lvert G\rvert$	group order, 1
$\beta_G(V)$	smallest upper bound for generating invariants, 2
V_d	binary forms of degree d, 2
T	Torus, 3
$X^\star(T)$	weights of T, 3
E	$X^\star(T)\otimes_{\mathbb{Z}}\mathbb{R}$, 3
$\chi_1,\ldots,\chi_n$	weights appearing in V, 3
$\mathcal{C}_V$	convex hull of $\chi_1,\ldots,\chi_n$, 3
$\mathrm{vol}(\mathcal{C}_V)$	volume of $\mathcal{C}_V$, 3
m	dimension of G, 3
r	rank of T, 3
N_V	nilpotency degree, 3
$\sigma(V)$	bound for primary invariants, 3
$\lVert\cdot\rVert$	norm on E, 3
L_V	$\max\{\lVert\chi_1\rVert,\lVert\chi_2\rVert,\ldots,\lVert\chi_n\rVert\}$, 3
$V/\!\!/G$	categorical quotient, 4
$\mathfrak{m}$	maximal homogeneous ideal, 4
$I_{\mathcal{N}}$	zero-fiber ideal, 4
$\mathcal{N}$	nullcone, 4
$\mathcal{R}$	Reynolds operator, 4
ψ	map $G\times V\to V\times V$ defined by $(g,v)\mapsto(v,g.v)$, 5
B	image of ψ, 5
$\overline{B}$	Zariski closure of B, 5
$\mathfrak{b}$	ideal in $\mathcal{O}(V)$ of $\overline{B}\subseteq V\times V$, 5
$\mathcal{R}_Y$	Reynolds operator $k[X,Y]\to k[X,Y]^G$, 5
Δ	diagonal map $V\to V\times V$, 5
$\Delta^\star$	algebra homomorphism $k[X,Y]\to k[X]$ corresponding to Δ, 5
p_1	projection $V\times V\to V$ onto the first factor, 6
$\mathfrak{p}$	homogeneous prime ideal, 6
$\mathfrak{p}_1$	linear part of $\mathfrak{p}$, 6

$\mathbb{P}(V \times V)$	projective space of $V \times V$, 7
$\alpha_1, \ldots, \alpha_{(m-r)/2}$	positive roots, 7
$\mathcal{W}$	Weyl group, 7
$e_1, \ldots, e_r$	Coxeter exponents, 7
$(\cdot, \cdot)$	$\mathcal{W}$-invariant scalar product, 7
$\hat{\gamma}$	dual element of $\gamma \in E$, 7
$\delta_G(V)$	number of $g \in G$ such that $\rho(g)$ lies in a given generic m-dimensional subspace of $\mathrm{End}(V)$, 7
P_i	polynomial on $G \times (V^\star \times V^\star)^{n+m-1}$, 8
$\bigwedge^n(V \times V)$	n-th antisymmetric power of $V \times V$, 9
G°	connected component of $\mathrm{id} \in G$, 10
I_G	ideal defining the group G as an affine variety, 11
Γ	graph of the action, 11
I_Γ	ideal defining Γ, 11
$\mathcal{G}_m$	multiplicative group, 12

Chapter II

$\mathbb{C}^\star$	multiplicative group $\mathbb{C} \setminus \{0\}$, 15
$F^\bullet R$	Filtration of a ring R, 16
$\mathrm{Gr}\, R$	associated graded ring, 16
gr	map $R \to \mathrm{Gr}\, R$, 16
deg	degree function, 16
D	locally nilpotent derivation, 17
$\mathcal{G}_a$	additive group $(\mathbb{C}, +)$, 17
$\dim R$	transcendence degree of the quotient field of R over $\mathbb{C}$, 17
$C(R)$	an invariant of rings, 17
v_x	valuation, 18

Chapter III

$\mathrm{Aut}_{\mathrm{hol}}(\mathbb{C}^n)$	group of holomorphic automorphisms of $\mathbb{C}^n$, 21
G	complex reductive group, 22
E	discrete subset $\subset \mathbb{C}^n$, 22
φ	non-rectifiable proper holomorphic embedding $\mathbb{C} \to \mathbb{C}^2$, 22
φ_k	non rectifiable proper holomorphic embedding $\mathbb{C}^{k+1} \to \mathbb{C}^{k+2}$, 22
$\mathcal{O}(Z)$	holomorphic functions on a Stein space Z, 23
$I_X(Y)$	Ideal of all functions in $\mathcal{O}(Y)$ vanishing on X, 23
ψ	map needed for definition of Rees space, 23
$\mathcal{R}(X, Y, Z, f)$	Rees space, 23
X^G	fixed point set, 25
$X/\!\!/G$	categorical quotient, 25
$h(x, y)$	holomorphic function on $\mathbb{C}^2$ vanishing on $\varphi(\mathbb{C})$, 25
H	subgroup of G, 27
V	representation of G, 27
a	homogeneous G-invariant on V, 27
$\mathfrak{g}$	Lie algebra of G, 28
T	maximal torus, 28
$\mathfrak{t}$	Lie algebra of T, 28

$s_1, \dots, s_r$	simple reflections, 28
$\alpha_1, \dots, \alpha_r$	simple roots, 28
$(\cdot, \cdot)$	Killing form, 28
$\mathcal{W}, \mathcal{W}'$	Weyl groups, 28
$V_{\mathbb{R}}$	real representation, 28
A_n	Alternating groups, 28
χ	character, 29
$\langle \cdot, \cdot \rangle$	scalar product for characters, 29
1_H	trivial character of H, 29
1_H^G	character of G induced by 1_H, 29
$\mathbb{R}[G/H]$	$\mathbb{R}$-space spanned by G/H seen as a G-module, 29
χ_H	restriction of χ to H, 29
B	Borel subgroup, 29
P_J	parabolic subgroup, 29
St	Steinberg character, 29
U	maximal unipotent subgroup, 29
323a	323 dimensional irreducible representation, 29
M_i	Mathieu groups, 30
J_1, J_2, J_3	Janko groups, 30
Suz	Suzuki group, 30
HS	Higman-Sims group, 30
McL	McLaughlin group, 30
Co_1, Co_2, Co_3	Conway groups, 30
He	Held group, 30
$Fi_{22}, Fi_{23}, Fi'_{24}$	Fischer groups, 30
$O'N$	O'Nan group, 30
Ly	Lyons group, 30
Ru	Rudvalis group, 30
HN	Harada-Norton group, 30
Th	Thompson group, 30
B	baby monster, 30
M	monster, 30
${}^3D_4(2):3, \dots$	for more group notation see the *Atlas of finite groups* ([9]), 30

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