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# Universal Hashing and Geometric Codes

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**Abstract.** We describe a new application of algebraic coding theory to universal hashing and authentication without secrecy. This permits to make use of the hitherto sharpest weapon of coding theory, the construction of codes from algebraic curves. We show in particular how codes derived from Artin-Schreier curves, Hermitian curves and Suzuki curves yield classes of universal hash functions which are substantially better than those known before.

**Keywords:** universal hashing; authentication; alebraic curves; Weierstraßpoint; orthogonal array; Reed-Solomon code; Suzuki code

## 1. Introduction

The concept of universal classes of hash functions was introduced by Carter and Wegman in [6]. It has found numerous applications. We mention cryptography, complexity theory, search algorithms and associative memory (see the Introduction in [15]). Most important are the following classes (see [16]).

*Definition 1.* Let $\epsilon > 0$. A multiset $\Sigma$ of $b$ functions from a $k$-set $C$ to a $v$-set $E$ is *$\epsilon$-almost strongly universal$_2$* (short: $ASU_2$) if

1. for every $u \in C$ and $x \in E$ the number of elements of $\Sigma$ mapping $u \mapsto x$ is $b/v$,
2. for every pair $u_1, u_2 \in C, u_1 \neq u_2$, and every pair $x_1, x_2 \in E$ the number of elements of $\Sigma$ affording the operation $u_1 \mapsto x_1, u_2 \mapsto x_2$ is $\leq \epsilon \cdot b/v$.

*Definition 2.* Let $\epsilon > 0$. A multiset $\Sigma$ of $b$ functions from a $k$-set $C$ to a $v$-set $E$ is *$\epsilon$-almost universal$_2$* (short: $AU_2$) if for every pair $u_1, u_2 \in C, u_1 \neq u_2$ the number $\delta(u_1, u_2)$ of elements $f \in \Sigma$ such that $f(u_1) = f(u_2)$ satisfies

$$\delta(u_1, u_2) \leq \epsilon \cdot b.$$

We use the language of *arrays*, just as in [3]. Thus a $(C, E)$-array $\Sigma$ consists of a set $C$ of columns, $|C| = k$, a set $E$ of entries, $|E| = v$, and is a multiset of cardinality $b$ of mappings $C \longrightarrow E$. The elements of $\Sigma$ are written as rows of the array.

The notion of $ASU_2$ is clearly a generalization of orthogonal arrays of strength 2. In fact, assume equality always holds in condition 2. of Definition 1. Then the number $\lambda$ of

elements of $\Sigma$ affording the operation $u_1 \mapsto x_1, u_2 \mapsto x_2$ is $\lambda = \epsilon \cdot b/v$. It is clear that $b = v^2\lambda$ in this case. Thus $\epsilon = 1/v$ and $\Sigma$ is an orthogonal array of strength 2. This description of $ASU_2$-classes as generalizations of orthogonal arrays of strength 2 has been observed in earlier work. It seems however to have escaped attention that $AU_2$-classes admit an even neater description: the columns of an $AU_2$-class of hash functions form the words of an error-correcting $v$-ary code. This is an equivalent description of $AU_2$-classes:

LEMMA 1 *Let $\epsilon > 0, |C| = k, |E| = v$, $\Sigma$ a $(C, E)$-array of $n$ functions from $C$ to $E$. Then the following are equivalent:*

- *$\Sigma$ is an $\epsilon - AU_2$ class of hash functions.*
- *The columns of $\Sigma$ form the words of a $v$-ary code of length $n$ with minimum distance $d$, where*

$$1 - \frac{d}{n} \leq \epsilon.$$

The proof is trivial, but the fact is important. The theory of $AU_2$-classes is nothing but the theory of error-correcting codes. This shows that the machinery of coding theory may be used to produce classes of hash functions. Of central importance is the composition-construction of Stinson's (see [16]).

*Definition 3.* Let $\Sigma_1$ be a $(C, E_1)$-array with $b_1$ and $\Sigma_2$ an $(E_1, E)$-array with $b_2$ rows. The *composition* $\Sigma = \Sigma_2 \circ \Sigma_1$ is the $(C, E)$-array with $b_1 \cdot b_2$ rows whose elements are all the compositions $f_2 \circ f_1$, where $f_1 \in \Sigma_1,\ f_2 \in \Sigma_2$.

In the new notation Stinson's composition-method for $ASU_2$-classes looks as follows:

THEOREM 1 (STINSON) *If $\Sigma_2$ is an $\epsilon_2 - ASU_2$ class with $k_1$ columns and $v$ entries and $\Sigma_1$ is an array whose columns form the words of a $k_1$-ary code of length $n$ with $k$ code words and minimum distance $d$ such that $\frac{d}{n} \geq 1 - \epsilon_1$, then the composition $\Sigma = \Sigma_2 \circ \Sigma_1$ is an $\epsilon - ASU_2$ with $k$ columns and $v$ entries, where*

$$\epsilon = \epsilon_1 + (1 - \epsilon_1)\epsilon_2 < \epsilon_1 + \epsilon_2.$$

The composition-construction needs two types of ingredients. For the $ASU_2$-classes we may use orthogonal arrays. We make use of a family constructed in [1], p. 363:

THEOREM 2 *Let $q$ be a prime-power and let $m, n$ be natural numbers, $m \geq n$. Then there is an*

$$OA_{q^{m-n}}(2, q^m, q^n).$$

In [16] the construction of this family is simplified. The present author extended the construction to general $t$ ([2]). Consider the second ingredient, the error-correcting code.

We fix the ground-field $\mathbb{F}_q$ and the relative minimum distance $d/n$ of a $q$-ary code. In fact the minimum distance has to be extremely large, as $\epsilon = 1 - \frac{d}{n}$ plays the role of a probability. For a fixed number $M$ of code-words we ask for the minimum length of such a code.

*Definition 4.* Let natural numbers $v$, $M$, and the real number $\epsilon$, $0 < \epsilon < 1$ be given. Define $m(\epsilon, v, M)$ as the minimum length $n$ of a $v$-ary code with $M$ codewords and minimum distance $d$ satisfying

$$d/n \geq 1 - \epsilon.$$

This is a somewhat unusual question in coding theory. Equally unusual is the fact that we are only interested in $q$-ary codes with relatively large $q$. Binary codes are not interesting at all in our context. Here is the corresponding notion for the $ASU_2$-classes:

*Definition 5.* Let natural numbers $v$, $M$, and the real number $\epsilon$, $0 < \epsilon < 1$ be given. Define $m_A(\epsilon, v, M)$ as the minimum number of functions of an $\epsilon - ASU_2$ class of hash functions from an $M$-set to a $v$-set.

With this terminology Stinson-composition looks as follows:

THEOREM 3 (STINSON-COMPOSITION)

$$m_A(\epsilon_1 + \epsilon_2, v, k) \leq m_A(\epsilon_1 + (1 - \epsilon_1)\epsilon_2, v, k) \leq m(\epsilon_1, k_1, k) \cdot m_A(\epsilon_2, v, k_1).$$

## 2. The Use of Geometric Codes

In order to make Stinson-composition efficient in the construction of $ASU_2$-classes of hash-functions, bounds on the function $m(\epsilon, q, q^k)$ are needed.

The monotonicity-properties of the function $m$ follow from the definition:

If $\epsilon_1 < \epsilon_2$, then $m(\epsilon_1, v, M) \geq m(\epsilon_2, v, M)$.

If $v_1 < v_2$, then $m(\epsilon, v_1, M) \geq m(\epsilon, v_2, M)$.

If $M_1 < M_2$, then $m(\epsilon, v, M_1) \leq m(\epsilon, v, M_2)$.

Here are some more basic properties of the function $m$:

$$m(\epsilon, v^i, M^i) \leq m(\epsilon, v, M).$$

This is Stinson's **Cartesian product-** construction (see [16]). In the case of linear codes

we may describe this as **extension of constants**. **Concatenation** of codes yields

$$m(\epsilon_1 + \epsilon_2, v, M) \leq m(\epsilon_2, v, M_1) \cdot m(\epsilon_1, M_1, M).$$

In [12] the relation between $ASU_2$-classes and $AU_2$-classes of hash functions is studied. We note the following main results in our present notation:

If $q$ is a prime-power, then

- $m(\epsilon, q, q(q-1)k + q) \leq m_A(\epsilon, q, k)$
- $m_A(\epsilon, q, (q-1)k + 1) \leq q \cdot m_A(\epsilon, q, k)$

The 2-dimensional Reed-Solomon code shows $m(\frac{1}{q}, q, q^2) \leq q$. Using the cartesian product-construction and concatenation of codes recursively one obtains

$$m\left(\frac{i}{q}, q, q^{2^i}\right) \leq q^i$$

for every prime-power $q$ and every $i \geq 1$. This is a construction of Stinson's, expressed in different words. It is optimal in case $i = 1$, as remarked by Stinson. We want to show how more sophisticated classes of linear codes, in particular of codes defined on algebraic curves, may be used to improve this bound considerably, for all $i > 1$. It is natural in our context to use the machinery of geometric codes in the following form:

THEOREM 4 (CANONICAL CONSTRUCTION) *Let $K$ be a function field of transcendence-degree 1 (equivalently: an algebraic curve) over the field $\mathbb{F}_q$ of constants, $P_0, P_1, \ldots P_n$ rational points (equivalently: prime divisors of degree 1) of $K$. Consider the divisors $D = P_1 + P_2 \ldots + P_n$, $G = uP_0$. Let $u_1 = 0, u_2, \ldots, u_l, \ldots$ be the pole-orders of $P_0$ in ascending order. Consider the code*

$$\mathcal{C}_l = \mathcal{C}(D, u_l P_0)$$

*of functions which are everywhere holomorphic except for a pole of degree $\leq u_l$ at $P_0$, evaluated at $P_1, \ldots P_n$ (this is the $L$-construction of [17]). Assume $u_l < n$. Then $\mathcal{C}_l$ has dimension $l$ and the following hold:*

- *$\mathcal{C}_l$ has minimum distance $\geq n - u_l$. Hence*

$$m\left(\frac{u_l}{n}, q, q^l\right) \leq n.$$

We need curves with many rational points and at least one rational Weierstraß point whose gaps are as large as possible.

In [14] and [13] a class $K_q^{(r)}$ of function fields defined over an arbitrary finite field $\mathbb{F}_q$ of constants is studied, where $r \geq 2$. Here $K_q^{(r)}$ is a tower of Artin-Schreier extensions of the rational function field. The following facts are to be found in [13]: The number $N_1$ of rational points of $K_q^{(r)}$ is $N_1 = q^r + 1$. There is a rational Weierstraß point $P_0$ whose semigroup of pole-orders is

$$\sum_{i=1}^{r} q^{r-i}(q+1)^{i-1}\mathbb{N}_0.$$

Let $\sum_{i=1}^{r} a_i q^{r-i}(q+1)^{i-1}$ be a pole-order of $P_0$, $a_i \geq 0$. Define for the moment the *weight* of this expression to be $w = \sum_{i=1}^{r} a_i$. It is easily seen that the weight $w$ of a pole-order is uniquely determined if $w < \frac{q}{r-1}$. It is also easily seen that under this assumption the representation of each pole-order of weight $w$ as a linear combination of the $q^{r-i}(q+1)^{i-1}$ is uniquely determined. The number of pole-orders of weight $w$ is then the number of distributions of $w$ undistinguishable objects in $r$ cells. It is a basic combinatorial fact that this latter number is

$$\binom{w+r-1}{r-1}.$$

Assume $i - 1 < \frac{q}{r-1}$. Upon using the well-known combinatorial identity

$$\sum_{j=m}^{n} \binom{j}{m} = \binom{n+1}{m+1}$$

we see that the number of pole-orders of weight $< i$ is precisely

$$\sum_{w=0}^{i-1} \binom{w+r-1}{r-1} = \binom{i+r-1}{r}.$$

If $\binom{i+r-1}{r} \geq 2^i$, then $u_{2^i} < iq^{r-1}$, hence

$$m\left(\frac{i}{q}, q, q^{2^i}\right) \leq q^r.$$

THEOREM 5 *Let $q$ be a prime-power, $i, r$ natural numbers such that*

$$q > (i-1)(r-1), \binom{i+r-1}{r} \geq 2^i.$$

*Then*

$$m\left(\frac{i}{q}, q, q^{2^i}\right) \leq q^r.$$

*Here are some examples:*

- *If* $q > 8$, *then*

$$m\left(\frac{5}{q}, q, q^{32}\right) \leq q^3.$$

- *If* $q \geq 23$, *then*

$$m\left(\frac{8}{q}, q, q^{256}\right) \leq q^4.$$

- *If* $q \geq 49$, *then*

$$m\left(\frac{12}{q}, q, q^{4096}\right) \leq q^5.$$

In fact we can get a precise asymptotic statement. Consider the binary entropy-function $H\colon (0,1) \longrightarrow \mathbb{R}_+$, where

$$H(x) = -x\log(x) - (1-x)\log(1-x).$$

Here the logarithm is binary. Observe $H(x) = H(1-x)$. The following inequality is well-known in information-theory, see [8]:

$$2^{mH(l/m)}/(m+1)^2 \leq \binom{m}{l} \leq 2^{mH(l/m)}.$$

Hence asymptotically

$$\binom{i+r-1}{r} \sim 2^{(i+r-1)H(r/(i+r-1))}.$$

We fix $i$ and ask for the smallest $r$ satisfying $\binom{i+r-1}{r} \geq 2^i$. Asymptotically this is equivalent to

$$(i+r-1)H\left(\frac{r}{i+r-1}\right) \geq i$$

and to

$$H\left(\frac{r}{i+r-1}\right) \geq \frac{i}{i+r-1}.$$

Put $x = \frac{r}{i+r-1}$. Then $\frac{i}{i+r-1} \sim 1-x$. Let $q_0 > 0$ be the unique solution of the equation

$$H(q_0) = q_0.$$

We may choose $r$ such that $x$ is arbitrarily close to $1-q_0$. Observe $r = (i-1)\cdot\frac{x}{1-x}$. We conclude:

THEOREM 6 *Let $q_0$ be the unique positive solution of the equation*

$$H(q_0) = q_0.$$

*For every $\epsilon > 0$ and sufficiently large $i$ we have*

$$m\left(\frac{i}{q}, q, q^{2^i}\right) \leq q^r,$$

*where $r = \lfloor (i-1)(1-q_0)/q_0 - \epsilon \rfloor$ and $q$ is an arbitrary prime-power, $q > (i-1)(r-1)$.*

The numerical values are

$$q_0 = .7729\ldots, (1-q_0)/q_0 \approx .2938$$

We note that the same number $q_0$ appears in the theory of **Sperner capacity,** a recently discovered extension of the concept of **Shannon capacity** of a graph. Denote by $L(n)$ the maximum cardinality of a family of pairs $(A_i, B_i)$ of subsets of an $n$-set satisfying

$$A_i \cup B_i \not\subseteq A_j \cup B_j \ (i \neq j),$$

$$A_i \cap B_j = \emptyset \text{ if and only if i=j.}$$

Then $L(n) \sim 2^{q_0 \cdot n}$ asymptotically in $n$. This is proved in [9].

For small values of $i$ and a quadratic ground-field we obtain improvements by means of **Hermitian codes**. Consider the **Hermitian curve** defined by the equation $X^{q+1} + Y^{q+1} + Z^{q+1} = 0$ over the field $\mathbb{F}_{q^2}$ of constants. This curve has genus $\binom{q}{2}$ and $q^3 + 1$ rational points. These form the well-known Hermitian unital. They are all Weierstraß points. The semigroup of pole-orders of any of them is $q\mathbb{N}_0 + (q+1)\mathbb{N}_0$. In particular

$$u_1 = 0, u_2 = q, u_3 = q+1, u_4 = 2q, \ldots u_8 = 3q+1, \ldots.$$

By choosing $l = 4$ and $l = 8$ in the canonical construction we get

$$m\left(\frac{2}{q^2}, q^2, q^8\right) \leq q^3,$$

$$m\left(\frac{3}{q^2} + \frac{1}{q^3}, q^2, q^{16}\right) \leq q^3$$

for every prime-power $q$.

THEOREM 7 *Let $q = p^{2f}$ be a quadratic prime-power. Then*

$$m\left(\frac{2}{q}, q, q^4\right) \leq q^{3/2}$$

$$m\left(\frac{3}{q} + \frac{1}{q^{3/2}}, q, q^8\right) \leq q^{3/2}.$$

Let us consider the distribution of pole-orders in greater detail. The $w+1$ integers between $w \cdot q$ and $w \cdot (q+1)$ are pole-orders. Let us call $w$ the **weight** of such a pole-order. If $w \leq q$, then a pole-order of weight $w$ doesn't have any smaller weight. The number of pole-orders of weight $\leq w$ is then $1 + 2 + \ldots + (w+1) = \binom{w+2}{2}$.

LEMMA 2 *Let $(u_l)$ be the pole-orders of the Hermitian curve over $\mathbb{F}_{q^2}$, in ascending order. If $w \leq q$, then*

$$u_{\binom{w+1}{2}+1} = w \cdot q$$

$$u_{\binom{w+2}{2}} = w \cdot (q+1).$$

In characteristic 2 we get further improvements by using a family of curves which admit the Suzuki groups as automorphism groups. This family is studied in [11]. Let $q = 2^{2f+1}$, $q_0 = 2^f$. The curve is defined over $\mathbb{F}_q$ by the homogeneous equation

$$X^{q_0}(Z^q + ZX^{q-1}) = Y^{q_0}(Y^q + YX^{q-1}),$$

has $q^2 + 1$ rational points, genus $q_0(q-1)$ and a Weierstraß point whose semigroup of pole-orders is

$$q\mathbb{N}_0 + (q+q_0)\mathbb{N}_0 + (q+2q_0)\mathbb{N}_0 + (q+2q_0+1)\mathbb{N}_0.$$

As before define the weight of the expression $a_1q + a_2(q+q_0) + a_3(q+2q_0) + a_4(q+2q_0+1)$ to be $w = \sum_i a_i$. Assume $w \leq q_0$. Then a pole-order of weight $w$ cannot have any smaller weight. Assume it can be written as a linear combination like above in more than one way:

$$\begin{aligned} &a_1q + a_2(q+q_0) + a_3(q+2q_0) + a_4(q+2q_0+1) \\ &\qquad = a_1'q + a_2'(q+q_0) + a_3'(q+2q_0) + a_4'(q+2q_0+1), \end{aligned}$$

where $\sum a_i = \sum a_i' = w \leq q_0$. Put $x_i = a_i - a_i'$. Then

$$x_1q + x_2(q+q_0) + x_3(q+2q_0) + x_4(q+2q_0+1) = 0$$

and the sum of the coefficients is zero, hence

$$x_2q_0 + 2x_3q_0 + x_4(2q_0 + 1) = 0.$$

It follows $q_0 \mid x_4$. We claim $x_4 = 0$.

Otherwise $x_4 = \pm q_0$, and this is possible only if $w = q_0$. We may assume without restriction $x_4 = q_0$, hence $a_4 = q_0, a_3 = a_2 = a_1 = 0, a_4' = 0$. Factoring out $q_0$ in the above equation yields

$$x_2 + 2x_3 + 2q_0 + 1 = 0.$$

As $x_2 = -a_2'$, $x_3 = -a_3'$ this shows $a_2' + 2a_3' = 2q_0 + 1$. Using the fact that $a_1' + a_2' + a_3' = q_0$, we get by subtraction $a_3' - a_1' = q_0 + 1$, a contradiction.

We have proved $x_4 = 0$. It follows

$$x_2 + 2x_3 = 0.$$

We conclude that the only ambiguity in the representation of pole-orders of weight $w \le q_0$ is the following: Coefficients $(a_1, a_2, a_3, a_4)$ and $(a_1 + x, a_2 - 2x, a_3 + x, a_4)$ represent the same pole-order ($x \in \mathbb{Z}$). The representation becomes unique by choosing $a_2 \in \{0, 1\}$. The same combinatorial fact used above shows that the number of pole-orders of weight $w \le q_0$ is $\binom{w+2}{2} + \binom{w+1}{2} = (w+1)^2$. Let $s_i = 1^2 + 2^2 + \ldots + i^2$. It is well-known that $s_i = (2i+1)(i+1)i/6$. It follows that the number of pole-orders of weight $\le w$ is $s_{w+1}$, if $w \le q_0$.

LEMMA 3 *Let $(u_l)$ be the pole-orders of the Suzuki curve over $\mathbb{F}_q$, $q = 2^{2f+1}$, in ascending order,*

$$s_i = 1^2 + 2^2 + \cdots + i^2 = (2i+1)(i+1)i/6.$$

*If $w \le q_0 = 2^f$, then*

$$u_{s_w+1} = w \cdot q$$

We record some low-dimensional cases, which yield improvements on Stinson's construction:

| $i$ | $s_i$ | bound |
|---|---|---|
| 2 | 5 | $m\left(\frac{2}{q}, q, q^6\right) \le q^2$ |
| 3 | 14 | $m\left(\frac{3}{q}, q, q^{15}\right) \le q^2$ |
| 4 | 30 | $m\left(\frac{4}{q}, q, q^{31}\right) \le q^2$ |
| 5 | 55 | $m\left(\frac{5}{q}, q, q^{56}\right) \le q^2$ |
| 6 | 91 | $m\left(\frac{6}{q}, q, q^{92}\right) \le q^2$ |
| 7 | 140 | $m\left(\frac{7}{q}, q, q^{141}\right) \le q^2$ |

In reality the situation is even better. For example there is not much of a difference between probabilities $\frac{3}{q^2}$ and $\frac{3}{q^2}+\frac{1}{q^3}$. For practical purposes the second statement of Theorem 7 should therefore be interpreted as

$$m\left(\approx \frac{3}{q}, q, q^8\right) \leq q^{3/2} \text{ (q a square prime-power)}.$$

The same situation occurs regularly when using the canonical construction. The Artin/Schreier-series yields examples:

$$m\left(\approx \frac{6}{q}, q, q^{64}\right) \leq q^3$$

$$m\left(\approx \frac{9}{q}, q, q^{512}\right) \leq q^4.$$

## 3. The Use of Deligne-Lusztig Codes for the Construction of Universal Hash Classes

The starting point in the Deligne-Lusztig theory of ordinary representations of finite groups of Lie type is the study of a certain variety, the Deligne-Lusztig variety, associated to a connected reductive algebraic group $G$ defined over a finite field. If $G$ has Lie rank 1, then the variety is a projective curve. Thus there are algebraic curves associated with the 2-dimensional linear groups, the 3-dimensional unitary groups, the Suzuki groups and the Ree groups (see [5], [7], [10]). The corresponding codes obtained via the $L$-construction (see Theorem 4) will be called **Deligne-Lusztig codes.** The case of the groups $PGL_2(q)$ leads to the Reed-Solomon codes. Thus the RS-codes are special cases of Deligne-Lusztig codes. The Hermitian codes (associated with the groups $PGU_3(q^2)$) and Suzuki codes (associated with ${}^2B_2(q)$, $q = 2^{2f+1}$) have been considered in the preceding section. In this section we will make use of these codes to construct $ASU_2$-classes of hash functions with a small number of keys (hash functions). It is natural to conjecture that also the Ree curves will yield good codes and good classes of hash functions. We have not yet been able to verify this as the Weierstraß-points and their pole orders seem to be unknown.

We use Stinson-composition (Theorem 1) with the class of orthogonal arrays of Theorem 2 in the role of $\Sigma_2$. In our notation this yields

$$m_A\left(\frac{1}{q^m}, q^m, q^n\right) \leq q^{n+m} \ (n \geq m, q \text{ a prime-power}).$$

Our aim is to get good upper bounds on $m_A(\frac{2}{q^m}, q^m, q^N)$, where $N$ is large with respect to $m$. This amounts to authenticating a source of $N \cdot \log_2 q$ bits using $m \cdot \log_2 q$ authenticator

bits, with deception probability $\epsilon \leq \frac{2}{q^m}$. The deception probability can never be smaller than the reciprocal of the number of authenticators. Moreover, in the case of equality the $ASU_2$-classes of hash functions is an orthogonal array of strength 2. In that case the number of keys is very large:

$$m_A\left(\frac{1}{q^m}, q^m, q^N\right) \geq q^N(q^m - 1) + 1 > q^N.$$

It was Wegman & Carter's crucial observation (see [18]) that the number of keys (of hash functions) can be dramatically reduced when $\epsilon$ is increased just a little bit. We consider the case $\epsilon = \frac{2}{q^m}$ when $\epsilon$ is the double of the theoretical minimum.

We use the canonical construction (Theorem 4) in its standard form. As we took care to determine the pole-orders of the Hermitian and Suzuki curves, it suffices to collect the data given in the preceding section. In the case of Reed-Solomon codes things are easier yet. The results are as follows:

THEOREM 8 *Let $q$ be a prime-power. Then the following holds:*

- $m(\frac{k-1}{q}, q, q^k) = q$ *for every* $k \geq 2$.
- *If* $w \leq q$, *then*

$$m\left(\frac{w}{q^2}, q^2, q^{2\{\binom{w+1}{2}+1\}}\right) \leq q^3$$

- *If* $q = 2^{2f+1}, q_0 = 2^f \geq w$, *then*

$$m\left(\frac{w}{q}, q, q^{1+(2w+1)(w+1)w/6}\right) \leq q^2.$$

Here the items correspond to Reed-Solomon, Hermitian and Suzuki codes. It follows from the Singleton bound that we have equality in the first case. We make use of Stinson-composition in the form of Theorem 3. Let $\mathbb{F}_{q^n}, n \geq m$ be the field over which the code is defined. As $\epsilon_2 = 1/q^m$ we have to take care that $\epsilon_1 \leq 1/q^m$. The result is as follows:

THEOREM 9 *Let $q$ be a prime-power, $m, n$ natural numbers. Then the following hold:*

1. $$m_A\left(\frac{2}{q^m}, q^m, q^{n(1+q^{n-m})}\right) \leq q^{2n+m}$$

   *($m \leq n$).*

2. $$m_A\left(\frac{2}{q^m}, q^m, q^{n\{1+q^{n-m}(q^{n-m}+1)/2\}}\right) \leq q^{5n/2+m}$$

   *($m \leq n \leq 2m$, $q^n$ a square).*

3. $$m_A\left(\frac{1}{2^{m-1}}, 2^m, 2^{n\{1+(2^{n-m+1}+1)(2^{n-m}+1)2^{n-m}/6\}}\right) \leq 2^{3n+m}$$

*($m \leq n \leq 2m-1$, $n$ odd).*

Let us compare the three methods (Reed-Solomon=RS, Hermite=H, Suzuki=S) of Theorem 9 in the binary case. We are thus looking for bounds on $m_A(\frac{1}{2^{m-1}}, 2^m, 2^N)$. Denote by log the binary logarithm, put $L = \log(N)$. When using the RS-method, we have to determine $n = n_{RS}$ from the equation $L = \log(n(1+2^{n-m}))$. This is approximately equivalent with $n_{RS} = L+m-\log(n_{RS})$. In the case of the H- and S-method we get $n_H \approx \frac{1}{2}L+m+\frac{1-\log(n_H)}{2}$ and $n_S \approx \frac{1}{3}L + m + \frac{\log 3-\log(n_S)}{3}$, respectively. Here $m \leq n_H \leq 2m, m \leq n_S \leq 2m-1$. The H-method needs less bits than the RS-method as soon as $4n_{RS} > 5n_H$. By substituting the above values of $n_{RS}$ and $n_H$ and putting $n_{RS} = \alpha \cdot n_H$ we see that this is equivalent with

$$L > \frac{2}{3}m + \log(n_H) + \frac{5}{3} + \frac{8}{3}\log(\alpha).$$

Certainly $\log(m) \leq \log(n_H) \leq 1 + \log(m)$. Furthermore we have a priori bounds on $\alpha$: in fact we are assuming $\alpha \geq 5/4$, and by equating the two expressions for $N$ we get $n_{RS}2^{n_{RS}-m} \approx n_H 2^{2(n_H-m)-1}$, equivalently $\alpha \approx 2^{(2-\alpha)n_H-m-1} \geq 5/4$. It follows $(2-\alpha)n_H - m - 1 \geq \log 5 > 2$, hence $1.25 \leq \alpha < 1.5$. We see that the H-method needs less key than the RS-method as soon as

$$L > \frac{2}{3}m + \log(m) + c_m,$$

where $c_m$ is a small constant, approximately $2.4 < c_m < 4.3$. The H-method is best as long as it is applicable, i.e. up to $L = 2m + \log(m)$. From then on the S-method is better.

What happens for even larger sources, when the S-method is no longer applicable in the form of Theorem 9, i.e. when $L > 3m + \log(m)$? If we use the Artin-Schreier curves $K_q^{(r)}$ as introduced before Theorem 5 in the same spirit as Deligne-Lusztig curves were used in Theorem 9, then let us speak of the $AS_r$-method, $r \geq 3$ (the case $r = 2$ is in fact not interesting). Moreover we can use geometric codes for arbitrarily large dimensions. Instead of the canonical construction of Theorem 4 we use then the equation $u_l = l + 1 + g$, where $g$ is the genus of the curve in question. It turns out that this variant is never best possible when applied to the Hermite curves or the $AS_r$-curves. In the case of Suzuki curves, let us call this the $S_+$-method. The following is the result of the (trivial) analysis when we compare all these methods in case $L > 3m + \log(m)$. Here we omit the summand $\log(m)$ as well as some small additive constant:

- If $3m \leq L \leq 14m$ (approximately), then the $S_+$-method is best.

- For $14m \leq L \leq 20m$ (approximately), method $AS_4$ is best.

- For larger $L$ method $AS_r (r > 4)$ will be best among the constructions considered here for approximately $(r-1)r \cdot m \leq L \leq r(r+1) \cdot m$.

We close with some explicit examples, which are applications of Theorem 9. Here $m = 20, q = 2$:

*Example 1:*

1. $m_A(2^{-19}, 2^{20}, 2^{2^{10}}) \leq 2^{72}$.

2. $m_A(2^{-19}, 2^{20}, 2^{2^{35}}) \leq 2^{110}$.

3. $m_A(2^{-19}, 2^{20}, 2^{2^{50}}) \leq 2^{125}$.

In the first case ($L = 10$), the RS-method is best. In fact $n_{RS} = 26, n_H = 22, n_S = 23$. The H-method needs $\frac{5}{2} \cdot 22 + 20 = 75$ bits, the S-method needs $3 \cdot 23 + 20 = 89$ bits.

In the second case ($L = 35$) we have $n_H = 36$, and the H-method is best. The RS- and S-method need 120 and 113 bits, respectively.

In case $L = 50$, the H-method is no longer applicable. We have $n_S = 35$. It follows that the S-method needs $3 \cdot 35 + 20 = 125$ bits, which is better than the RS-method ($n_{RS} = 64$, needs 148 bits).

We restate the last inequality in a more familiar language: It is possible to authenticate a $2^{50}$-bit source with 20 authenticator bits and 125 bits of key such that the probabilities of success of an impersonation- or substitution-attack are bounded by $2^{-19}$.

Finally let us consider the practicality of the construction. For simplicity we concentrate on the case of Example 1,1. The source has size $2^{2^{10}}$. The use of Stinson composition implies that our family of hash functions is the composition of a family $A$ of functions: $2^{2^{10}} \longrightarrow 2^{26}$ und a family $B$ of functions: $2^{26} \longrightarrow 2^{20}$. For $B$ we took the $2^{46}$ rows of the orthogonal array $OA_{64}(2, 2^{26}, 2^{20})$ of Theorem 2. More explicitly the elements of $B$ are pairs $(\beta, \gamma)$, where $\beta \in \mathbb{F}_{2^{26}}, \gamma \in \mathbb{F}_{2^{20}}$. The entry in row $(\beta, \gamma)$, and column $x \in \mathbb{F}_{2^{26}}$ of the orthogonal array is $\Phi(\beta \cdot x) + \gamma$, where $\Phi$ is a fixed $\mathbb{F}_2$-linear mapping: $\mathbb{F}_{2^{26}} \longrightarrow \mathbb{F}_{2^{20}}$.

For $A$ we took the coordinate functions of the 40-dimensional $RS$-code over $\mathbb{F}_{2^{26}}$. Thus each element of $A$ is an element $\alpha \in \mathbb{F}_{2^{26}}$, and each source state is a polynomial $p(X)$ of degree $\leq 39$ with coefficients in $\mathbb{F}_{2^{26}}$. The hash functions are triples $(\alpha, \beta, \gamma)$, where $\alpha, \beta \in \mathbb{F}_{2^{26}}, \gamma \in \mathbb{F}_{2^{20}}$. Thus $26 + 26 + 20 = 72$ random bits are needed to choose a hash function uniformly at random. The corresponding hashed value is

$$\Phi(p(\alpha) \cdot \beta) + \gamma \in \mathbb{F}_{2^{20}}.$$

We need field arithmetic in $\mathbb{F}_{2^{26}}$, but only addition in $\mathbb{F}_{2^{20}}$. The field $\mathbb{F}_{2^{26}}$ should be seen as an

extension of $\mathbb{F}_2$. The mapping $\Phi$ may be chosen as projection onto the first 20 coordinates, say.

## Conclusion

We have seen that the theory of almost universal ($AU_2-$) classes of hash functions is equivalent with the theory of error-correcting codes. The mechanism of geometric codes imposes precise conditions on algebraic curves to yield $AU_2-$ classes of hash functions which can be efficiently used as one of the two ingredients in Stinson's composition-construction for almost strongly universal ($ASU_2-$) classes of hash functions. Most importantly we saw that Deligne-Lusztig curves and certain Artin-Schreier curves allow the construction of $ASU_2-$ classes of hash functions which use much less key space than the methods which had hitherto been used.

Part of the material of this paper was used in a joint publication with T. Johansson, G. Kabatianskii and B. Smeets [4] for CRYPTO 93. In particular the method of Theorem 9 is inspired by the use my coauthors in [4] make of Reed-Solomon codes.

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