A SHORT PROOF OF AN IDENTITY OF SYLVESTER

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ABSTRACT. We present two short proofs of an identity found by Sylvester, and rediscovered by Louck. The first proof is an elementary version of Knuth's proof, and is analogous to Macdonald's proof of a related identity of Milne. The second is Sylvester's own proof of his identity.
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1. Sylvester's identity

Our purpose in this paper is to present two proofs of a fundamental identity found in Sylvester's work, which is, in Sylvester's own words [17, p. 90], " a simple theorem for expressing, by means of partial fractions, the sum of the homogeneous powers and products of any number of quantities ".

The identity in question is

$$h_{q-n+1}(\mathbf{x}) = \sum_{r=1}^{n} x_r^q \prod_{\substack{i=1\\ i\neq r}}^{n} \frac{1}{(x_r - x_i)}, \tag{1.1}$$

where q is a non-negative integer, and the complete homogeneous symmetric function $h_m(\mathbf{x})$ in the variables $\mathbf{x} \equiv (x_1, \dots, x_n)$ is defined by means of the generating function

$$\sum_{m\geq 0} h_m(\mathbf{x})\ t^m = \prod_{i=1}^{n} \frac{1}{1 - x_i t}. \tag{1.2}$$

Further, if $m < 0$, then $h_m(\mathbf{x})$ is defined to be 0.

Sylvester [16, p. 42] uses the fact that the sum in (1.1) is 0 when $q = 0, \dots, n-2$, and is a polynomial when $q \geq n-1$. In his later work on partitions [17], he uses (1.1) again. But the identity is most clearly formulated only in the lectures he gave in 1859, see [18, p. 156]. A little more than a hundred years later, (1.1) was rediscovered by Louck [8]. Chen and Louck [2] have pointed out that for $q = 0, 1, \dots, n-1$, the identity was known to Waring [20] in 1779. The $q = n-1$ case of (1.1) was rediscovered by Good [4] in his elegant proof of Dyson's [3] conjecture.

In §2, we will present two short proofs of Sylvester's theorem, both involving partial fraction expansions. The first proof succeeds in finding the left hand side of (1.1) by starting from the sum in the right hand side, and is analogous to Macdonald's proof of a related identity. The second is Sylvester's own proof of his identity, and, as suggested by his description above, transforms the

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left hand side of (1.1) into the sum on the right. Finally, in §3, we will comment briefly upon the importance of these identities.

2. Partial Fractions

We first consider Macdonald's clever proof of an identity found by Milne [11]:

$$\sum_{r=1}^{n}(1-y_r)\prod_{\substack{i=1\\i\neq r}}^{n}\left[\frac{1-x_iy_i/x_r}{1-x_i/x_r}\right]=1-y_1y_2\cdots y_n. \tag{2.1}$$

Macdonald (see [13]) proved (2.1) by setting $t=0$ in the partial fraction expansion

$$\prod_{i=1}^{n}\frac{(1-tx_iy_i)}{(1-tx_i)}=y_1\cdots y_n+\sum_{r=1}^{n}\frac{1-y_r}{1-tx_r}\prod_{\substack{i=1\\i\neq r}}^{n}\left[\frac{1-x_iy_i/x_r}{1-x_i/x_r}\right]. \tag{2.2}$$

Our first proof of Sylvester's identity is analogous to Macdonald's proof of (2.1). To prove (1.1), we consider the partial fraction expansion

$$z^q\prod_{i=1}^{n}\frac{1}{(z-x_i)}=\sum_{r=1}^{n}\frac{x_r^q}{z-x_r}\prod_{\substack{i=1\\i\neq r}}^{n}\frac{1}{(x_r-x_i)}+p_q(z), \tag{2.3}$$

where $p_q(z)=0$ if $q=0,1,\dots,n-1$. Further, if $q\geq n$, then it is a polynomial of degree $q-n$. Let $F_q(x_1,\dots,x_n)$ represent the sum on the right hand side of (1.1). Next, set $z=0$ in (2.3) to obtain

$$F_{q-1}(x_1,\dots,x_n)=p_q(0).$$

Our proof will be complete once we compute $p_q(0)$. But $p_q(0)$ is nothing but the constant term in the quotient obtained when z^q is divided by $\prod_{i=1}^{n}(z-x_i)$. That is,

$$\begin{aligned}p_q(0)=&\text{ the constant term in } z^q\prod_{i=1}^{n}\frac{1}{(z-x_i)}\\ =&\text{ the coefficient of } z^{n-q} \text{ in } \prod_{i=1}^{n}\frac{1}{(1-x_i/z)}\\ =&\, h_{q-n}(\mathbf{x}),\end{aligned}$$

by comparing with (1.2). It follows that

$$F_q(x_1,\dots,x_n)=p_{q+1}(0)=h_{q-n+1}(\mathbf{x}).$$

This finishes the derivation of Sylvester's identity.

Macdonald's proof of (2.1) is very simple, but the choice of the particular rational function on the left hand side of (2.2) is unmotivated. A similar remark holds for (2.3). However, a simple observation remedies this situation.

Consider once again the sum side of (1.1), where n is replaced by $n+1$, and x_{n+1} is renamed z. In this manner, we obtain

$$F_q(x_1,\dots,x_n,z)=\sum_{r=1}^{n}\frac{x_r^q}{x_r-z}\prod_{\substack{i=1\\i\neq r}}^{n}\frac{1}{(x_r-x_i)}+z^q\prod_{i=1}^{n}\frac{1}{(z-x_i)}. \tag{2.4}$$

It is clear that (2.4) is the same as (2.3), our starting point in the proof of Sylvester's identity. The particular choice of the rational function considered is now transparent. The same observation applies to Macdonald's proof of Milne's identity.

This observation is also relevant to Askey's proof of Milne's identity, which is reproduced by Milne [13]. Askey first proved that the sum side of (2.1) is independent of $x_1, \dots, x_n$. Suppressing even the dependence on $y_1, \dots, y_n$, we let f_n denote the left hand side of (2.1). To complete his proof, Askey found a simple recursion for f_n:

$$f_{n+1} = y_{n+1} f_n + (1 - y_{n+1}),$$

from which (2.1) follows quite easily.

Instead, we find another recursion for f_n, by replacing n by $n+1$ in (2.1), and taking the limit as $x_{n+1} \to 0$. In this manner, we obtain

$$f_{n+1} - f_n = y_1 \dots y_n - y_1 \dots y_{n+1}. \tag{2.5}$$

We also have the initial condition $f_1 = 1 - y_1$. Milne's identity follows by noting that

$$\begin{aligned} f_1 + \sum_{r=1}^{n-1} (f_{r+1} - f_r) &= 1 - y_1 + \sum_{r=1}^{n-1} (y_1 \dots y_r - y_1 \dots y_{r+1}) \\ &= 1 - y_1 \cdots y_n, \end{aligned}$$

by telescoping. Recursion (2.5) is perhaps even simpler than Askey's recursion.

The proof of (1.1) presented above is also related to Sylvester's proof of his identity. In Sylvester's notes [18], where (1.1) appears explicitly, he does not include his proof. But based on his remarks reproduced above, and some of his work in his previous paper [17], it seems likely that he obtained (1.1), by considering the partial fraction expansion

$$\prod_{i=1}^{n} \frac{1}{z - x_i} = \sum_{r=1}^{n} \frac{1}{z - x_r} \prod_{\substack{i=1 \\ i \neq r}}^{n} \frac{1}{(x_r - x_i)}. \tag{2.6}$$

By equating the coefficients of z^{-q-1} on both sides of the equation, we immediately obtain (1.1). Compare this with our computation of $p_q(0)$ above.

It is interesting to note that setting $z = 0$ in (2.6), and replacing x_i by x_i^{-1}, we obtain Good's identity, the $q = n - 1$ case of (1.1).

3. Concluding Remarks

Our first proof of Sylvester's identity is an elementary version of the proof given by Knuth [7, §1.2.3, problem 33], who found it necessary to use Cauchy's residue theorem. Variations of Sylvester's proof are given by Chen and Louck [2] and Strehl and Wilf [15], though these authors prefer to use the Lagrange interpolation formula rather than partial fractions. Knuth mentions that special cases of (1.1) are useful in the theory of divided differences. Indeed, (1.1) has been rediscovered by Verde-Star [19] in this context. It appears in the context of mathematical physics in the work of Louck and Biedenharn [9, 10]. Far reaching generalizations of (1.1) have been found by Gustafson and Milne [6] and by Chen and Louck [2].

Milne [11] first proved (2.1) using (1.1). Several other proofs of (2.1), including those of Macdonald and Askey, are compiled by Milne [13]. Yet another proof is given by Strehl and Wilf [15]. Identity (2.1) is fundamental in the study of multiple basic hypergeometric series, see for instance Milne [12, 13] and Gustafson [5].

Finally, we note that Macdonald's proof of (2.1) is also relevant. Bhatnagar and Milne [1] and Schlosser [14] have used (2.2) to generalize identities which Milne [12] found using (2.1).

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