Number of different degree sequences of a graph with no isolated vertices

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Example
Example

7 different degree sequences
A partition is a weakly decreasing sequence of non-negative integers, where almost all numbers are zero.

\[ \lambda = 3, 3, 2, 2, 1, 1, 0, \ldots \]
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The weight of a partition is the sum over this sequence.

\[ |\lambda| = 12 \]
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The weight of a partition is the sum over this sequence.

\[ |\lambda| = 12 \]

The length of a partition is the number of nonzero parts.

\[ l(\lambda) = 6 \]
Ferrers Diagram

Partitions are visualized by left adjusted boxes in the first quadrant.
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Conjugate Partition

The *conjugate* partition $\lambda'$ is the sequence of numbers of boxes in the columns.
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```
  6
  4
  2
```
The *conjugate* partition $\lambda'$ is the sequence of numbers of boxes in the columns.

$6, 4, 2 = (3, 3, 2, 2, 1, 1)'$
A partition $\lambda$ is called *graphical*, if there is a simple (undirected, no loops, no multi-edges) graph whose vertex degree sequence equals $\lambda$. 
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- graphical partitions only exist for even weight
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- Graphical partitions only exist for even weight
- Not all even weight partitions are graphical
Example
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Number of different degree sequences of a graph with no isolated vertices – p.7/21
Example

\[
\begin{array}{c}
3 \\
2
\end{array}
\begin{array}{c}
3 \\
2
\end{array}
\begin{array}{c}
3 \\
2
\end{array}
\begin{array}{c}
3 \\
1
\end{array}
\begin{array}{c}
1
\end{array}
\]

\[3,3,2,2,1,1\]
Problem

We want

\[ g(n) := \text{number of graphical partitions of length } n. \]
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For general partitions only useful in the case of a maximal size of parts \((< n)\)
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$$g(n) := \text{number of graphical partitions of length } n.$$  

For general partitions only useful in the case of a maximal size of parts ($< n$)
Durfee square $= (2, 2)$
Durfee

Durfee square = \((2, 2)\)
Durfee size = 2
Durfee Decomposition
Durfee Decomposition
Durfee Decomposition

\[ L = (4, 2) \]
\[ R = (2, 2) \]
Dominance Order

The 'natural' partial order on partitions. Let \( \mu, \nu \) be two partitions

\[
\mu \geq \nu : \iff \forall k \geq 1 : \sum_{i=1}^{k} \mu_i \geq \sum_{i=1}^{k} \nu_i
\]
Dominance Order

The 'natural' partial order on partitions. Let $\mu$, $\nu$ be two partitions

\[ \mu \trianglerighteq \nu : \iff \forall k \geq 1 : \sum_{i=1}^{k} \mu_i \geq \sum_{i=1}^{k} \nu_i \]

Dominance order is compatible with graphical partitions:

\[ \nu \text{ graphical, } \mu \trianglerighteq \nu \Rightarrow \mu \text{ graphical} \]
Theorem:
A partition $\lambda$ of even weight is graphical

$\Leftrightarrow$

$L(\lambda) \supseteq R(\lambda)$
Recursion Formula (1)

\[ G(n) := \text{set of graphical partitions of length } n \]
\[ G_s(n) := \text{set of graphical partitions of length } n \]
\[ \text{and maximal part of size } s \]

\[ G(n) = G_1(n) \cup \ldots \cup G_{n-1}(n) \]
Recursion Formula (1)

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and maximal part of size \( s \)

\[ G(n) = G_1(n) \cup \ldots \cup G_{n-1}(n) \]

Each \( G_s(n) \) is decomposed into disjoint subsets according to the weight

\[ G_s(n) = G_{s,2}(n) \cup \ldots \cup G_{s,n^*}(n-1)(n) \]
Recursion Formula (2)

Each set $G_{s,w}(n)$ is decomposed according to the size of the Durfee square

$$G_{s,w}(n) = G_{s,w,1}(n) \cup \ldots \cup G_{s,w,n-1}(n)$$
Recursion Formula (2)

Each set $G_{s,w}(n)$ is decomposed according to the size of the Durfee square

$$G_{s,w}(n) = G_{s,w,1}(n) \cup \ldots \cup G_{s,w,n-1}(n)$$

From the Durfee decomposition and the criterion we get a bijection:

$$G_{s,w,d}(n) \leftrightarrow \{ (\mu, \nu) \ with \ 1 \leq l(\mu) \leq d, \mu_1 = n - d, l(\nu) = id \}.$$ 

$$|\nu| + |\mu| = n - (d - 1) \ast d$$
Recursion Formula (3)

\[ P(s_1, l_1, w_1, l_2, w_2) := \text{pairs } (\mu, \nu) \text{ with } \]
\[ \mu \succeq \nu, \mu_1 = s_1 \]
\[ l(\mu) = l_1, |\mu| = w_1 \]
\[ l(\nu) = l_2, |\nu| = w_2 \]
Recursion Formula (3)

\[ P(s_1, l_1, w_1, l_2, w_2) := \text{pairs} \ (\mu, \nu) \text{ with} \]
\[ \mu \geq \nu, \ \mu_1 = s_1 \]
\[ l(\mu) = l_1, |\mu| = w_1 \]
\[ l(\nu) = l_2, |\nu| = w_2 \]

rewrite above recursion with \( r = n - (d - 1) \times d \):

\[ G_{s,w,d}(n) \longleftrightarrow \bigcup_{j = 1, \ldots, d} P(n - d, j, l, d, r - l) \]
\[ l = 0, \ldots, r \]
Recursion Formula (4)
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\[ P(s_1, l_1, w_1, l_2, w_2) \]
Recursion Formula (4)

\[ P(s_1, l_1, w_1, l_2, w_2) \]

\[ \bigcup_{i = 0, \ldots, l_1} \bigcup_{j = 0, \ldots, l_2} P(s_1 - 1, i, w_1 - l_1, j, w_2 - l_2) \]
Product Formula

We count pairs $\mu \geq \nu$, with certain properties
We count pairs $\mu \triangleright \nu$, with certain properties
Unique minimal partition $\mu^-$, unique maximal partition $\nu^+$. 
Product Formula

We count pairs $\mu \trianglerighteq \nu$, with certain properties
Unique minimal partition $\mu^-$, unique maximal partition $\nu^+$.
If $\mu^- \trianglerighteq \nu^+$ then (with $p(\ldots) = |P(\ldots)|$)

$$p(s_1, l_1, w_1, l_2, w_2) = p(s_1, l_1, w_1, 0, 0) \left( \sum_{i=1, \ldots, w_2-l_2+1} p(i, l_2, w_2, 0, 0) \right).$$
### Results

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<td>$g(20), \ldots$</td>
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<td>7429.160296</td>
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### Results

<table>
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<th>$g(28),\ldots, g(34)$</th>
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Concluding Remarks

Limiting factors:

memory to store intermediate results

time if you do not store intermediate results
References

- Sierksma, Hoogeveen: Seven Criteria for Integer Sequences being Graphic, J. Graph Theory, 1991.
- N. Sloane: online database of integer sequences, number A095268