

Number of Graphical Partitions

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Partition

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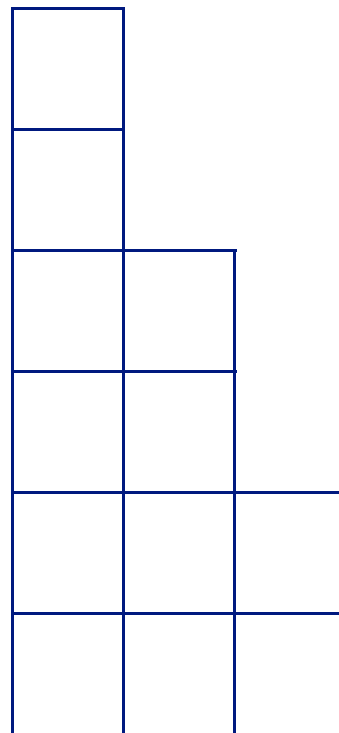
$$|\lambda| = 12$$

The **length** of a partition is the number of nonzero parts.

$$l(\lambda) = 6$$

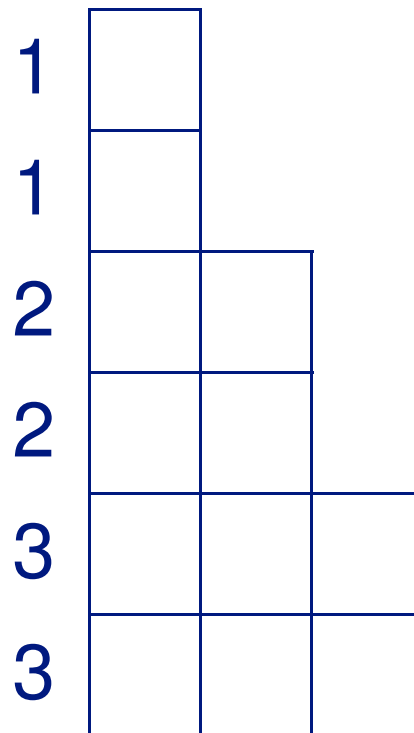
Ferrers Diagram

Partitions are visualized by left adjusted boxes in the first quadrant.



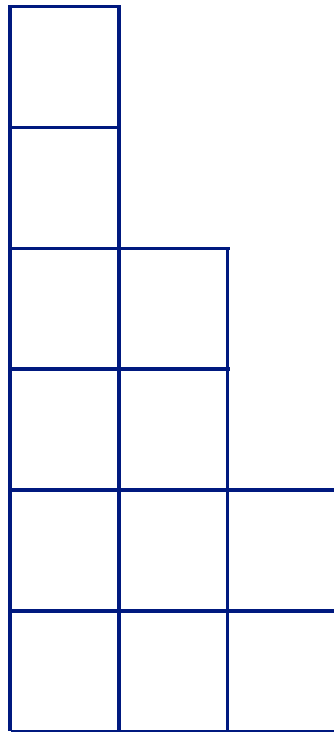
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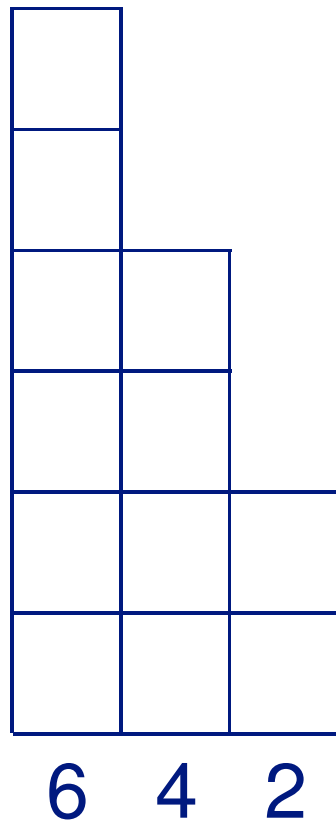
Conjugate Partition

The conjugate partition λ' is the sequence of numbers of boxes in the columns.



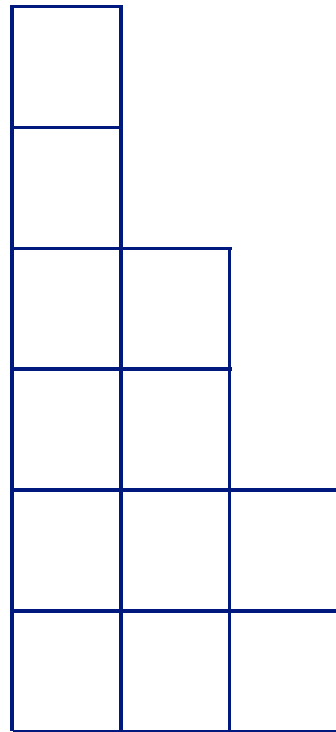
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$$6 \quad 4 \quad 2 = (3, 3, 2, 2, 1, 1)'$$

Graphical Partitions

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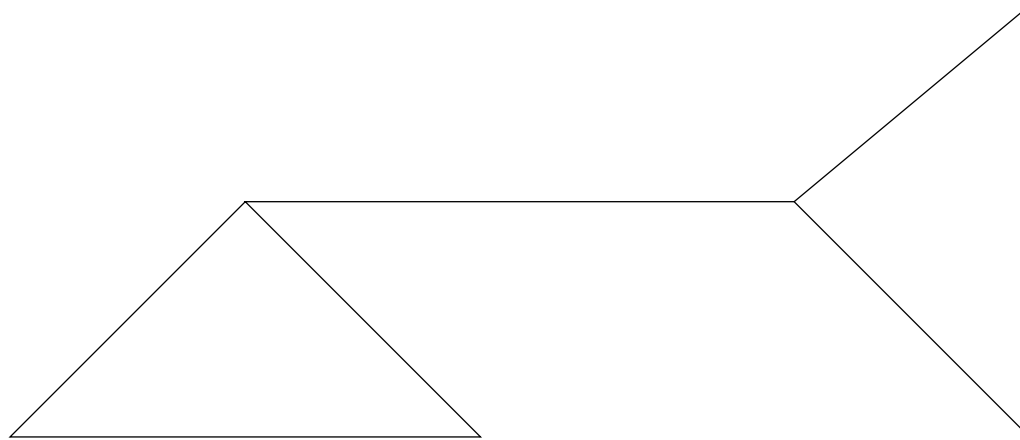
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Graphical Partitions

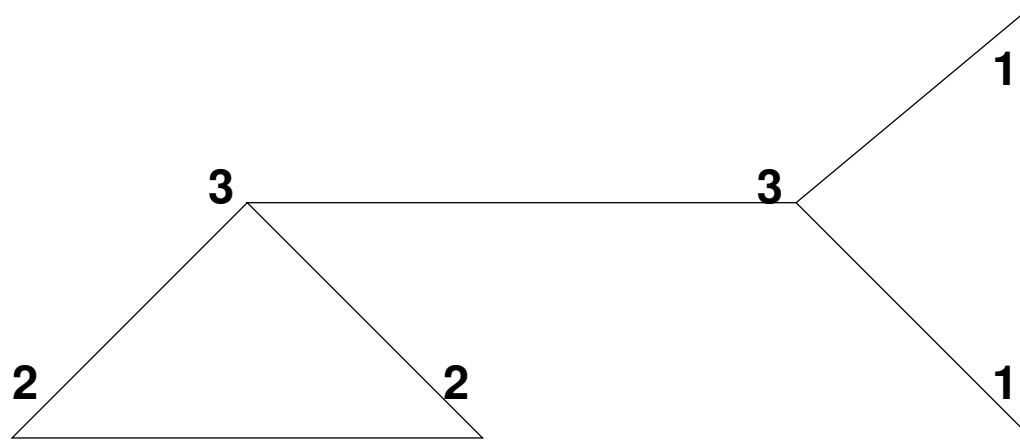
A partition λ is called **graphical**, if there is a simple (undirected, no loops, no multi-edges) graph whose vertex degree sequence equals λ .

- graphical partitions only exist for even weight
- not all even weight partitions are graphical

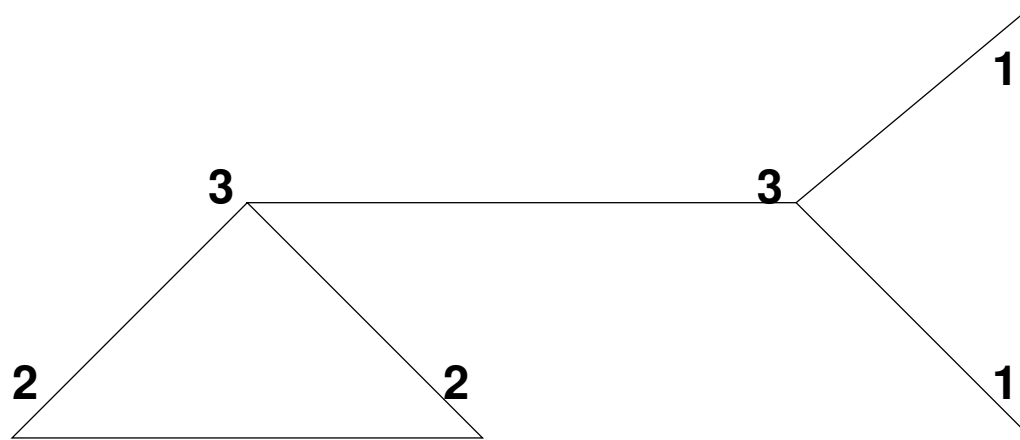
Example



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3, 3, 2, 2, 1, 1

Open Questions

The number $g(n)$ of graphical partitions of weight n is smaller than $p(n)$, the number of partitions.

$$\lim_{n \rightarrow \infty} \frac{g(n)}{p(n)} = ?$$

Known: $\lim < 0.25$

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For the number of partitions there is a generating function, which allows the fast computation of $p(n)$.

For $g(n)$: missing

Criteria for being Graphical

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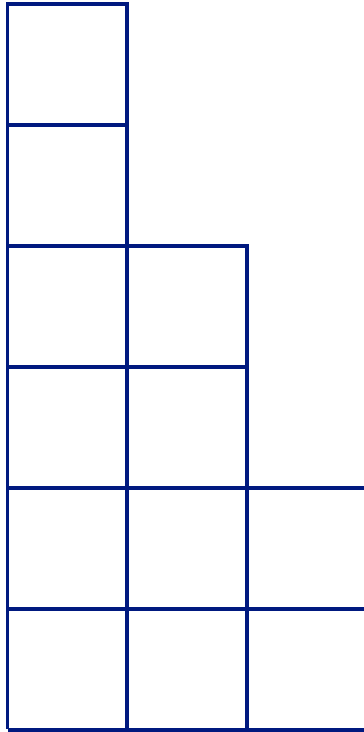
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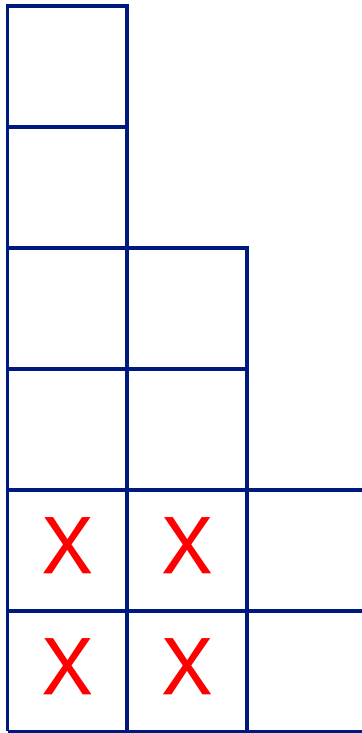
- Hässelbarth: for all $k \leq \text{Durfee size}$

$$\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k (\lambda'_i - 1)$$

Durfee

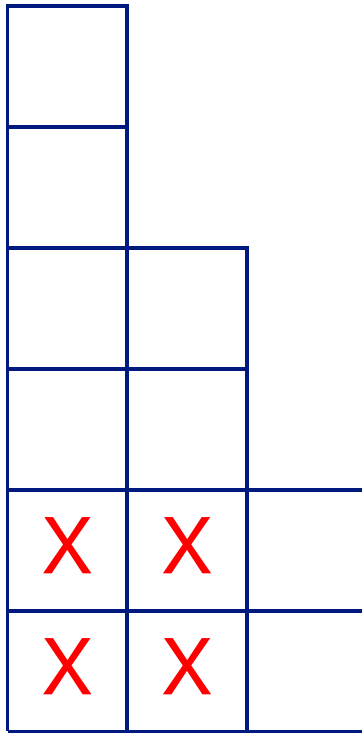


Durfee



Durfee square = $(2, 2)$

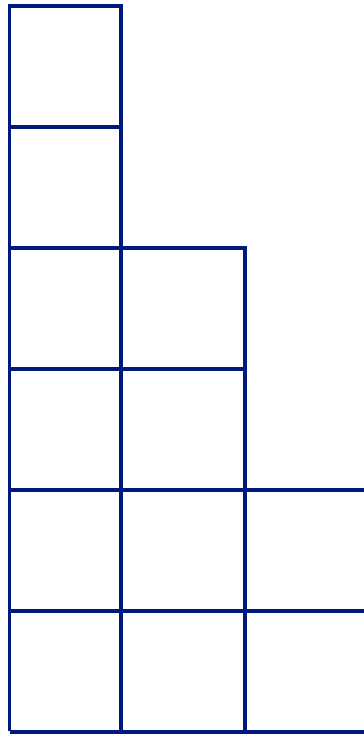
Durfee



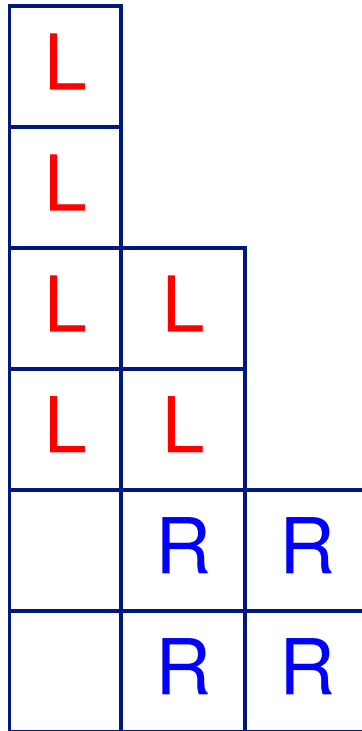
Durfee square = $(2, 2)$

Durfee size = 2

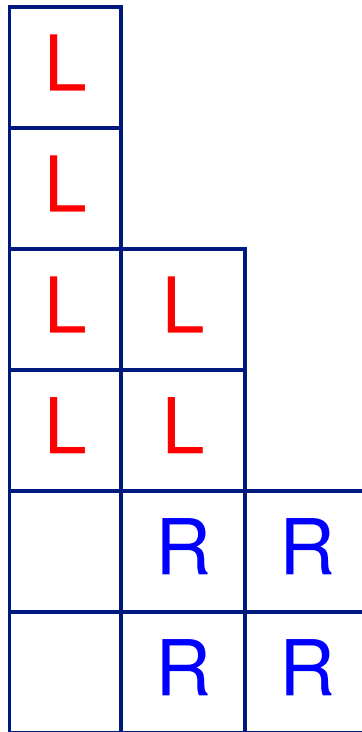
Durfee Decomposition



Durfee Decomposition



Durfee Decomposition



$$L = (4, 2)$$

$$R = (2, 2)$$

Dominance Order

The 'natural' order on partitions.

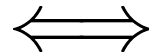
Let μ, ν be two partitions

$$\mu \succeq \nu \Leftrightarrow \forall k \geq 1 : \sum_{i=1}^k \mu_i \geq \sum_{i=1}^k \nu_i$$

New Criterion

Theorem

A partition λ of even weight is graphical



$$L(\lambda) \supseteq R(\lambda)$$

Recursion Formula (1)

$G(n)$:= set of graphical partitions of weight n

$G_i(n)$:= set of graphical partitions of weight n
and Durfee size i

$$G(n) = G_1(n) \dot{\cup} \dots \dot{\cup} G_{\lceil \sqrt{n} \rceil}(n)$$

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From the Durfee decomposition a bijection:

$$G_i(n) \longleftrightarrow \left\{ (\mu, \nu) \text{ with } \begin{array}{l} \mu \triangleright \nu \\ l(\mu) \leq i, l(\nu) = i \\ |\nu| + |\mu| = n - (i - 1) * i \end{array} \right\}.$$

Recursion Formula (2)

$P(m, k, n, l) :=$ pairs of partitions (μ, ν) with

$$\mu \supseteq \nu$$
$$l(\mu) = k, |\mu| = m$$
$$l(\nu) = l, |\nu| = n$$

Recursion Formula (2)

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rewrite above recursion with $r = n - (i - 1) * i$:

$$G_i(n) \longleftrightarrow \dot{\bigcup}_{\substack{j = 1, \dots, i \\ s = 0, \dots, r}} P(s, j, r - s, i)$$

Recursion Formula (3)

L	L		
L	L		
L	L	L	
L	L	L	
		R	R
		R	R
		R	R

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$$P(m, k, n, l) \rightsquigarrow \dot{\cup}_{\substack{i = 0, \dots, k \\ j = 0, \dots, l}} P(m - k, i, n - l, j)$$

$$p(m, k, n, l) = \sum_{\substack{i = 0, \dots, k \\ j = 0, \dots, l}} p(m - k, i, n - l, j)$$

Telescoping Sum

For $m > n$:

$$p(m, k, n, l) = \sum_{\substack{i = 0, \dots, k \\ j = 0, \dots, l}} p(m - k, i, n - l, j)$$

Telescoping Sum

For $m > n$:

$$\begin{aligned} p(m, k, n, l) &= \sum_{\substack{i = 0, \dots, k \\ j = 0, \dots, l}} p(m - k, i, n - l, j) \\ &= p(m - 1, k - 1, n, l) \\ &\quad + p(m, k, n - 1, l - 1) \\ &\quad - p(m - 1, k - 1, n - 1, l - 1) \\ &\quad + p(m - k, k, n - l, l) \end{aligned}$$

Results

n	$g(n)$	$p(n)$	$g(n)/p(n)$
100	69065657	190569292	.3624175
200	1.397805.210533	3.972999.029388	. 3518262
220	7.443670.977177	21.248279.009367	. 3503187

Barnes, Savage 1995

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1000	7.812520.197904 .651287.725407.239942	24.061467.864032 .622473.692149.727991	.3246900

Concluding Remarks

Limiting factors:

memory to store intermediate results
(18GB for $n=1000$)

time if you do not store intermediate results

Thank you very much for your attention.

